General Relativity: Solutions to exercises in Lecture I

January 22, 2018

Exercise 1

Consider a binary system of gravitating objects of masses M and m.

- First consider the case in which $m \ll M$ and where the small-mass object is in quasi-circular orbit around the more massive object. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [*Hint: use a co-ordinate system centred in M*].
- Now let $m = M$ and the binary be in circular orbit around the Newtonian centre of mass of the system. Draw the trajectory in two-space and the worldline in a $1+1$- and in a $2+1$-dimensional spacetime [*Hint: use a co-ordinate system centred in the Newtonian centre of mass*].

Solution 1

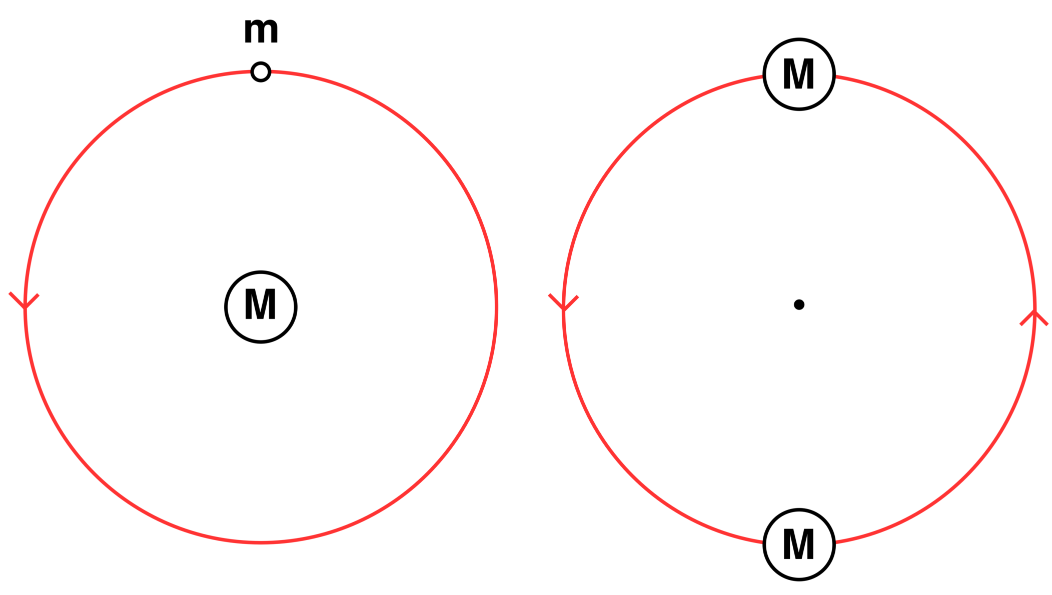

Figure 1: Trajectories in two-space for the cases $m \ll M$ (left) and $m = M$ (right).

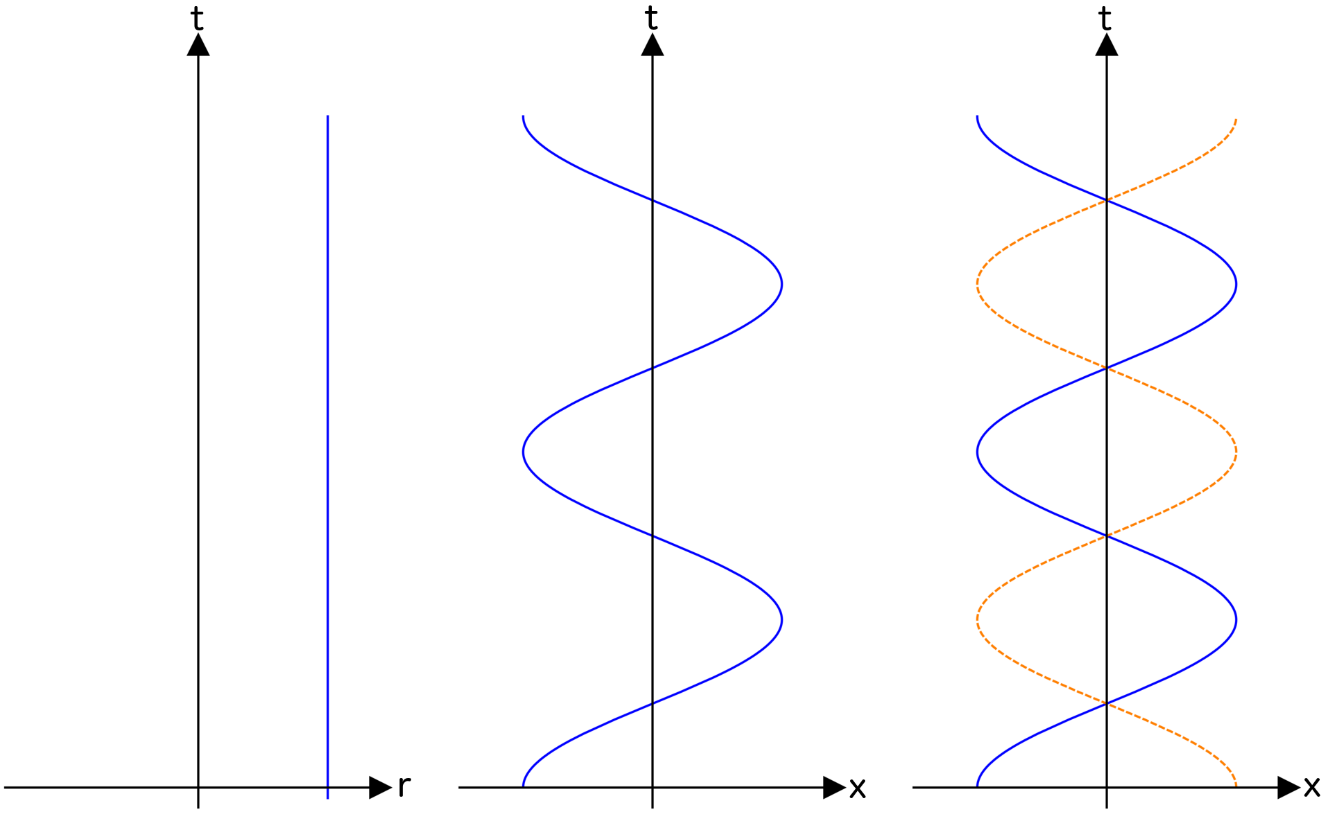

Figure 2: Worldline in 1 + 1-dimensional spacetime for the case $m \ll M$ in polar co-ordinates (left panel) and Cartesian co-ordinates (middle panel), and for the case $m = M$ in Cartesian co-ordinates (right panel).

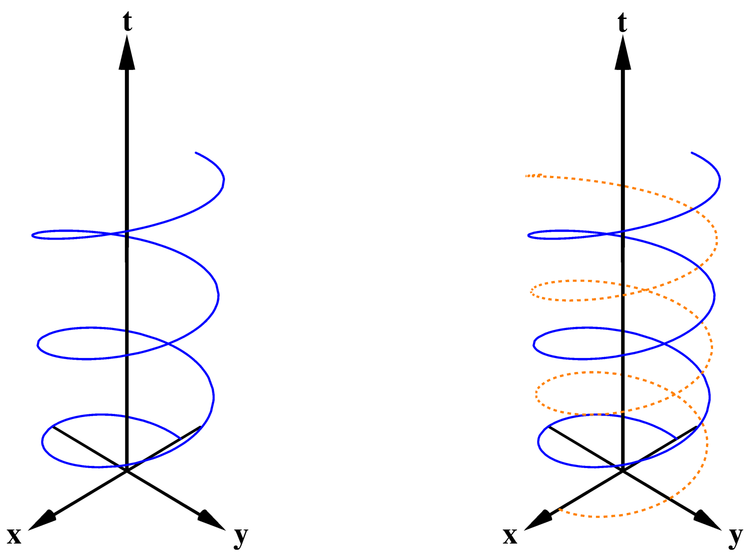

Figure 3: Worldline in 2 + 1-dimensional spacetime for the case $m \ll M$ (left) and $m = M$ (right).

Exercise 2

Consider a two-dimensional space and cover it with two co-ordinate maps: a Cartesian map where $\{x^\mu\} = (x, y)$ and a polar map where $\{x^{\mu'}\} = (r, \theta)$.

- Find the co-ordinate transformation $\boldsymbol{f}\colon\ x^\mu \to x^{\mu'}$
- Find the inverse co-ordinate transformation $\boldsymbol{f}^{-1}\colon\ x^{\mu'} \to x^\mu$
- Find the components of the transformation matrix $\Lambda^{\mu'}{}_{\mu}$ and its determinant $J' := \left|\partial x^{\mu'}/\partial x^\mu\right|$
- Find the components of the inverse transformation matrix $\Lambda^{\mu}{}_{\mu'}$ and its determinant $J := \left|\partial x^{\mu}/\partial x^{\mu'}\right|$
- Show that $\Lambda^{\mu'}{}_{\mu}\,\Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}$ and that $J\,J' = 1$

Solution 2

The co-ordinate transformation is given by:

$$\boldsymbol{f} : \begin{cases} r = \left(x^2 + y^2\right)^{1/2} \\ \theta = \arctan\left(y/x\right) \end{cases} \tag{1}$$

The inverse co-ordinate transformation is given by:

$$\boldsymbol{f}^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \tag{2}$$

The transformation matrix is given by:

$$\begin{aligned}
\Lambda^{\mu'}{}_{\mu} &= \frac{\partial x^{\mu'}}{\partial x^\mu} \\
&= \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} \\
&= \begin{pmatrix} x\left(x^2+y^2\right)^{-1/2} & y\left(x^2+y^2\right)^{-1/2} \\ -y\left(x^2+y^2\right)^{-1} & x\left(x^2+y^2\right)^{-1} \end{pmatrix} &(3)\\
&\equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{1}{r}\sin\theta & \dfrac{1}{r}\cos\theta \end{pmatrix} \ , &(4)
\end{aligned}$$

and its determinant is given by:

$$\begin{aligned}
J' &= \left|\partial x^{\mu'}/\partial x^\mu\right| \\
&= \frac{x^2}{\left(x^2+y^2\right)^{3/2}} + \frac{y^2}{\left(x^2+y^2\right)^{3/2}} \\
&= \left(x^2+y^2\right)^{-1/2} \ , &(5)
\end{aligned}$$

or alternatively, using equation (4), is given by:

$$\begin{aligned} J' &= \frac{\cos^2\theta}{r} + \frac{\sin^2\theta}{r} \\ &= \frac{1}{r} \ . \end{aligned} \tag{6}$$

It is trivial to confirm that both expressions for J' are equivalent.

The inverse transformation matrix is given by:

$$\begin{aligned} \Lambda^{\mu}{}_{\mu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \\ &= \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} && (7) \\ &\equiv \begin{pmatrix} x\left(x^2+y^2\right)^{-1/2} & -y \\ y\left(x^2+y^2\right)^{-1} & x \end{pmatrix} \ , && (8) \end{aligned}$$

and its determinant is given by:

$$\begin{aligned} J &= \left|\partial x^{\mu}/\partial x^{\mu'}\right| \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r \ , \end{aligned} \tag{9}$$

or alternatively, using equation (8), is given by

$$\begin{aligned} J &= \frac{x^2}{\left(x^2+y^2\right)^{1/2}} + \frac{y^2}{\left(x^2+y^2\right)^{1/2}} \\ &= \left(x^2+y^2\right)^{1/2} \ . \end{aligned} \tag{10}$$

It is again trivial to confirm that both expressions for J are equivalent.

Matrix multiplication of equations (3) and (8) or equations (4) and (7) yields the identity matrix, confirming the result $\Lambda^{\mu'}{}_{\mu}\,\Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}$. It is also straightforward to confirm that $J\,J' = 1$ in both co-ordinate systems.

Exercise 3

Consider a three-dimensional space and cover it with two co-ordinate maps: a Cartesian one where $\{x^{\mu}\} = (x, y, z)$ and a polar one where $\{x^{\mu'}\} = (r, \theta, \phi)$. Address all of the questions in Exercise 2.

Solution 3

The co-ordinate transformation is given by:

$$\boldsymbol{f} : \begin{cases} r = \left(x^2 + y^2 + z^2\right)^{1/2} \\ \theta = \arccos\left[z\left(x^2 + y^2 + z^2\right)^{-1/2}\right] \\ \phi = \arctan\left(y/x\right) \end{cases} \tag{11}$$

The inverse co-ordinate transformation is given by:

$$\boldsymbol{f}^{-1} : \begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \tag{12}$$

The transformation matrix is given by:

$$\begin{aligned} \Lambda^{\mu'}_{\ \mu} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \\ &= \begin{pmatrix} \partial r/\partial x & \partial r/\partial y & \partial r/\partial z \\ \partial\theta/\partial x & \partial\theta/\partial y & \partial\theta/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{pmatrix} \\ &= \begin{pmatrix} x\left(x^2+y^2+z^2\right)^{-1/2} & y\left(x^2+y^2+z^2\right)^{-1/2} & z\left(x^2+y^2+z^2\right)^{-1/2} \\ \dfrac{xz}{\left(x^2+y^2\right)^{1/2}\left(x^2+y^2+z^2\right)} & \dfrac{yz}{\left(x^2+y^2\right)^{1/2}\left(x^2+y^2+z^2\right)} & -\dfrac{\left(x^2+y^2\right)^{1/2}}{x^2+y^2+z^2} \\ y\left(x^2+y^2\right)^{-1} & x\left(x^2+y^2\right)^{-1} & 0 \end{pmatrix} \qquad (13) \\ &\equiv \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \dfrac{1}{r}\cos\theta\cos\phi & \dfrac{1}{r}\cos\theta\sin\phi & -\dfrac{1}{r}\sin\theta \\ \dfrac{\sin\phi}{r\sin\theta} & \dfrac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} , \qquad (14) \end{aligned}$$

whereby, upon simplification, its determinant may be found as:

$$\begin{aligned} J' &= \left|\partial x^{\mu'}/\partial x^{\mu}\right| \\ &= \left(x^2+y^2\right)^{-1/2}\left(x^2+y^2+z^2\right)^{-1/2} \ , \end{aligned} \tag{15}$$

or alternatively, using equation (14), is given by:

$$J' = \frac{1}{r^2\sin\theta} \ . \tag{16}$$

It is once more trivial to confirm that both expressions for J' are equivalent.

The inverse transformation matrix is given by:

$$\begin{aligned}
\Lambda^{\mu}{}_{\mu'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \\
&= \begin{pmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial \phi \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial \phi \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial \phi \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} && (17) \\
&\equiv \begin{pmatrix} x\left(x^2+y^2+z^2\right)^{-1/2} & xz\left(x^2+y^2\right)^{-1/2} & -y \\ y\left(x^2+y^2+z^2\right)^{-1/2} & yz\left(x^2+y^2\right)^{-1/2} & x \\ z\left(x^2+y^2+z^2\right)^{-1/2} & -\left(x^2+y^2\right)^{1/2} & 0 \end{pmatrix} , && (18)
\end{aligned}$$

whereby, upon simplification, its determinant may be found as:

$$J = r^2 \sin\theta \ , \tag{19}$$

or alternatively, using equation (18), is given by:

$$J = \left(x^2+y^2\right)^{1/2}\left(x^2+y^2+z^2\right)^{1/2} \ . \tag{20}$$

As before, it is again trivial to confirm that both expressions for J are equivalent.

As in Exercise 2, Matrix multiplication of equations (13) and (18) or equations (14) and (17) yields the identity matrix, confirming the result $\Lambda^{\mu'}{}_{\mu}\,\Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}$. It is also straightforward to confirm that $J\,J' = 1$ in both co-ordinate systems.

General Relativity: Solutions to exercises in Lecture II

January 22, 2018

Exercise 1

Consider two co-ordinate systems in a two dimensional space $\{x^\mu\} = (x, y)$ and $\{x^{\mu'}\} = (r, \theta)$ which are related through the well-known co-ordinate transformation

$$\boldsymbol{f} : \begin{cases} r = (x^2 + y^2)^{1/2} \\ \theta = \arctan(y/x) \end{cases}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Discuss the differences between the transformation matrix employed to transform a covector in this space

$$\left(\tilde{\mathrm{d}}x\right)_\mu = \Lambda^{\mu'}{}_\mu \left(\tilde{\mathrm{d}}x\right)_{\mu'} \ , \tag{1}$$

and the one employed in the co-ordinate transformation

$$x^{\mu'} = \Lambda^{\mu'}{}_\mu \, x^\mu \ . \tag{2}$$

Solution 1

The matrix involved in the transformation of the gradient $\left(\tilde{\mathrm{d}}x\right)_\mu = \Lambda^{\mu'}{}_\mu \left(\tilde{\mathrm{d}}x\right)_{\mu'}$ is different from the matrix used in the transformation $x^{\mu'} = \Lambda^{\mu'}{}_\mu \, x^\mu$. The two matrices, although written identically, are in fact transposes of each other.

To illustrate this, consider the co-ordinate systems $\{x^\mu\} = (x, y)$ and $\{x^{\mu'}\} = (r, \theta)$. It follows that $x^1 = x$, $x^2 = y$; $x^{1'} = r$, $x^{2'} = \theta$. One may now calculate the transformation between co-ordinate systems as:

$$\begin{aligned} x^{1'} &= r = \Lambda^{1'}{}_\mu \, x^\mu \\ &= \Lambda^{1'}{}_1 \, x^1 + \Lambda^{2'}{}_2 \, x^2 \\ &= \frac{\partial x^{1'}}{\partial x^1} x^1 + \frac{\partial x^{1'}}{\partial x^2} x^2 \\ &= \frac{\partial r}{\partial x} x + \frac{\partial r}{\partial y} y \ , \end{aligned} \tag{3}$$

and similarly $x^{2'} = (\partial\theta/\partial x)x + (\partial\theta/\partial y)y$. We may now write the transformation matrix as:

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \partial r/\partial x & \partial r/\partial y \\ \partial\theta/\partial x & \partial\theta/\partial y \end{pmatrix} . \tag{4}$$

On the other hand, for $\left(\tilde{\mathrm{d}}x\right)_{\mu} = \Lambda^{\mu'}{}_{\mu}\left(\tilde{\mathrm{d}}x\right)_{\mu'}$, consider the following explicit transformation:

$$\begin{aligned} \left(\tilde{\mathrm{d}}x\right)_{1} &= \Lambda^{\mu'}{}_{1}\left(\tilde{\mathrm{d}}x\right)_{\mu'} \\ &= \Lambda^{1'}{}_{1}\left(\tilde{\mathrm{d}}x\right)_{1'} + \Lambda^{2'}{}_{1}\left(\tilde{\mathrm{d}}x\right)_{2'} \\ &= \frac{\partial r}{\partial x}\left(\tilde{\mathrm{d}}x\right)_{1'} + \frac{\partial\theta}{\partial x}\left(\tilde{\mathrm{d}}x\right)_{2'} . \end{aligned} \tag{5}$$

Similarly, one finds $\left(\tilde{\mathrm{d}}x\right)_{2} = (\partial r/\partial y)\left(\tilde{\mathrm{d}}x\right)_{1'} + (\partial\theta/\partial y)\left(\tilde{\mathrm{d}}x\right)_{2'}$. The transformation matrix may now be written as:

$$\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \partial r/\partial x & \partial\theta/\partial x \\ \partial r/\partial y & \partial\theta/\partial y \end{pmatrix} \tag{6}$$

$$= \chi^{\mu'}{}_{\mu} . \tag{7}$$

Clearly $\left(\chi^{\mu'}{}_{\mu}\right)^{\mathrm{T}} = \Lambda^{\mu'}{}_{\mu}$ from equation (4), i.e. the transformation matrices are transposes of each other, as required.

Exercise 2

Consider two co-ordinate systems in a four-dimensional spacetime $x^{\mu} = (t, x, y, z)$ and $x^{\mu'} = (u, v, y, z)$ that are related through the co-ordinate transformation

$$\boldsymbol{f} : \begin{cases} u = t - x \\ v = t + x \end{cases}$$

and its inverse

$$\boldsymbol{f}^{-1} : \begin{cases} t = \frac{1}{2}(v + u) \\ x = \frac{1}{2}(v - u) \end{cases}$$

- Compute the matrices employed in the transformations

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, x^{\mu} \qquad\qquad x^{\mu} = \Lambda^{\mu}{}_{\mu'}\, x^{\mu'} . \tag{8}$$

- Consider a four-vector with components $U^{\mu} = (1, 0, 0, 0)^{\mathrm{T}}$ in the co-ordinate system x^{μ} and compute the new components $U^{\mu'}$ in the co-ordinate system $x^{\mu'}$.
- Repeat the calculation for the new vector $V^{\mu} = (-1/2, 1/2, 0, 0)^{\mathrm{T}}$. Interpret the results.

Solution 2

For the first part of the question, computing the transformation matrices, first consider ${\Lambda^{\mu'}}_{\mu}$.

$$\begin{aligned} \Lambda^{0'}_{\mu} &= \frac{\partial x^{0'}}{\partial x^{\mu}} \\ &= \frac{\partial u}{\partial x^{\mu}} \ , \end{aligned} \tag{9}$$

from which one obtains the following non-zero components:

$$\begin{aligned} {\Lambda^{0'}}_{0} &= \frac{\partial u}{\partial t} = 1 \ , && (10) \\ {\Lambda^{0'}}_{1} &= \frac{\partial u}{\partial x} = -1 \ . && (11) \end{aligned}$$

Similarly,

$$\begin{aligned} {\Lambda^{1'}}_{\mu} &= \frac{\partial x^{1'}}{\partial x^{\mu}} \\ &= \frac{\partial v}{\partial x^{\mu}} \ , \end{aligned} \tag{12}$$

from which one obtains the following non-zero components:

$$\begin{aligned} {\Lambda^{1'}}_{0} &= \frac{\partial v}{\partial t} = 1 \ , && (13) \\ {\Lambda^{1'}}_{1} &= \frac{\partial v}{\partial x} = 1 \ . && (14) \end{aligned}$$

Finally, one may also show that the remaining non-zero components of ${\Lambda^{\mu'}}_{\mu}$ are

$$\begin{aligned} {\Lambda^{2'}}_{2} &= 1 \ , && (15) \\ {\Lambda^{3'}}_{3} &= 1 \ . && (16) \end{aligned}$$

The transformation matrix may now be written as

$${\Lambda^{\mu'}}_{\mu} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{17}$$

For the inverse transformation matrix we follow the same procedure, from which the inverse transformation matrix is found as

$${\Lambda^{\mu}}_{\mu'} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{18}$$

The second part of the question asks to calculate U^μ in the new co-ordinate system, i.e. $U^{\mu'}$. Whilst it is obvious that one can do this through matrix multiplication, consider instead the following:

$$\begin{aligned} U^{\mu'} &= \Lambda^{\mu'}_{\ \ \mu} U^\mu \\ &= \Lambda^{\mu'}_{\ \ 0} U^0 \\ &= \Lambda^{\mu'}_{\ \ 0} \ , \end{aligned} \tag{19}$$

where the fact that the only non-zero component of U^μ is U^0 has been used. One can then read directly from equation (17) the solution as

$$U^\mu = (1, 1, 0, 0)^{\mathrm{T}} \ . \tag{20}$$

For the third and final part of this question one can again apply matrix multiplication to obtain the result, or consider the basis components as follows:

$$\begin{aligned} V^{\mu'} &= \Lambda^{\mu'}_{\ \ \mu} V^\mu \\ &= \Lambda^{\mu'}_{\ \ 0} V^0 + \Lambda^{\mu'}_{\ \ 1} V^1 \ . \end{aligned} \tag{21}$$

Considering this term by term yields

$$\begin{aligned} V^{0'} &= \Lambda^{0'}_{\ \ 0} V^0 + \Lambda^{0'}_{\ \ 1} V^1 \\ &= (1).(-1/2) + (-1).(1/2) \\ &= -1 \ , \end{aligned} \tag{22}$$

and

$$\begin{aligned} V^{1'} &= \Lambda^{1'}_{\ \ 0} V^0 + \Lambda^{1'}_{\ \ 1} V^1 \\ &= (1).(-1/2) + (1).(1/2) \\ &= 0 \ , \end{aligned} \tag{23}$$

from which it immediately follows that

$$V^{\mu'} = (-1, 0, 0, 0)^{\mathrm{T}} \ . \tag{24}$$

The second part may be interpreted as follows. In $\{x^\mu\}$ the four-vector U^μ represents a particle at rest, since all spatial components are zero: the particle may be represented as a vertical worldline in a $1+1$-spacetime. However, when transforming to $\{x^{\mu'}\}$ one finds that $U^{\mu'}$ has two non-zero components, implying that the particle no longer appears stationary and is moving with a constant velocity. Represented as a worldline in a $1+1$-spacetime (u, v) the worldline would be a line of constant positive (and finite) gradient.

For the third part, the vector V^μ has non-zero spatial components and so has a velocity of -1 in the x-direction. Represented as a worldline in a $1+1$-spacetime (t, x) it would be represented by a line of constant, finite and non-zero gradient. However, when transformed into $\{x^{\mu'}\}$, the four-vector $V^{\mu'}$ has zero spatial components. So in the co-ordinate system $\{x^{\mu'}\}$ the four-vector V^μ appears stationary.

Exercise 3

Consider a 1 + 1 representation of the sub-spaces with two co-ordinate systems (t, x) and (u, v).

- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := \mathrm{d}x/\mathrm{d}t = 0$.
- Draw in the two spacetimes the worldline of a particle with velocity $\dot{x} := k$ $(x = kt)$ with $k < 1$.
- Interpret the results.

Solution 3

In this question it is assumed we use the co-ordinate transformations as defined in Exercise 2.

For the first part, let us term the first particle as particle A. Since $\dot{x}_A = 0$ this implies $x_A = \text{const}$. The particle is stationary and at rest in the (t, x) co-ordinate system. In the (u, v) co-ordinate system one may write

$$u_A = t - x_A \; , \tag{25}$$
$$v_A = t + x_A \; , \tag{26}$$

from which one may conclude

$$\frac{u_A}{v_A} = \frac{t - x_A}{t + x_A} < 1. \tag{27}$$

Since $\partial u_A/\partial v_A \simeq (t - x_A)/(t + x_A) = (v_A - 2x_A)/v$. Integrating this yields

$$u(v) = v - \frac{2x_A}{v^2} + \text{const.} \tag{28}$$

We may set the integration constant to zero without loss of generality. We may now plot equation (28) for various values of x_A, the case of $x_A = 0$ corresponding to a straight line of constant gradient 1. The worldlines in both co-ordinate systems are illustrated in Figure 1 by the solid blue line.

For the second part of this question let us term the second particle as particle B. For particle B one has $\dot{x}_B = k$ (i.e. $x_B = kt$), where $k < 1$. The particle is now moving with constant velocity k and can be represented as a worldline of gradient $k < 1$ in the (t, x) co-ordinate system. In the (u, v) co-ordinate system one may write

$$u_B = t - kt = t(1 - k) \; , \tag{29}$$
$$v_B = t + kt = t(1 + k) \; , \tag{30}$$

from which one may conclude

$$\frac{u_B}{v_B} = \frac{1 - k}{1 + k}. \tag{31}$$

The condition that $(1 - k)/(1 + k) > 0$ implies that $|k| < 1$. Considering values of k in this range, the following condition on the gradient of the worldline may be obtained

$$\frac{u_B}{v_B} = \frac{1 - k}{1 + k} \begin{cases} < 1 \text{ if } k > 0 \text{ (Case B)} \; , \\ > 1 \text{ if } k < 0 \text{ (Case B}') \; . \end{cases}$$

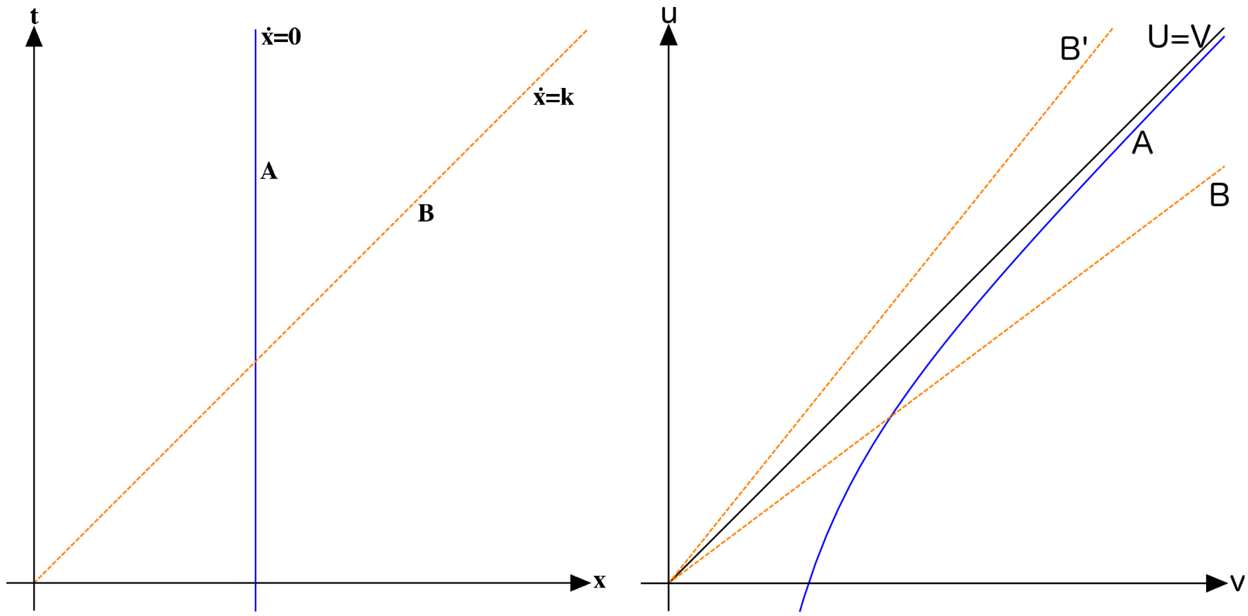

Figure 1: Worldliness for particles A and B in the (t, x) co-ordinate system (left) and the (u, v) co-ordinate system (right).

The worldlines in both co-ordinate systems are illustrated in Figure 1 by the dashed orange line.

For the final part of the question, for particle A, in the (t, x) co-ordinate system it is at rest. However, in the (u, v) co-ordinate system it is moving with constant velocity. For particle B, consider the limit $k \to 1$, whereby $\partial x/\partial t = 1$ and $\partial u_B/\partial v_B = 0$. In the (x, y) co-ordinate system the particle is moving with constant velocity, but in the limit $k \to 1$, in the (u, v) co-ordinate system this implies that the particle appears stationary (or the (v, u) co-ordinate system depending on how one labels the axes).

General Relativity: Solutions to exercises in Lecture III

January 22, 2018

Exercise 1

Consider $\boldsymbol{T}$ as a contravariant tensor of rank 2 with components $T^{\mu\nu}$. Under what conditions can this tensor be cast as the product of two contravariant vectors $\mathbf{U}$ and $\mathbf{V}$, i.e. such that $T^{\mu\nu} = U^\mu V^\nu$?

Solution 1

In a given basis $\boldsymbol{T}$ is represented by a matrix $T^{\mu\nu}$. In these terms a necessary and sufficient condition to enable $T^{\mu\nu}$ to be written as $T^{\mu\nu} = U^\mu V^\nu$ is that all columns of the matrix $T^{\mu\nu}$ must be proportional to each other (linearly dependent). As an example, consider the following matrix:

$$T^{\mu\nu} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 8 & 16 \\ 3 & 6 & 12 & 24 \\ 4 & 8 & 16 & 32 \end{pmatrix} .$$

Since the columns of this matrix are proportional to one another, we may choose $U^\mu = (1, 2, 3, 4)$ and $V^\nu = (1, 2, 4, 8)$, thus satisfying $T^{\mu\nu} = U^\mu V^\nu$.

Let us now consider this in a co-ordinate independent (covariant) way. $T^{\mu\nu} = U^\mu V^\nu$ if and only if $S^\mu = T^{\mu\nu} x_\nu$ is in the same direction, for any given x_ν.

Consider the set of orthonormal basis vectors $\mathbf{e}^0$, $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$ which by definition must satisfy $\mathbf{e}^\mu \mathbf{e}_\nu = \delta^\mu{}_\nu$. The direction of S^μ is independent of the choice of x_ν (by linearity) if and only if it is independent of our basis vectors $\mathbf{e}^0$, $\mathbf{e}^1$, $\mathbf{e}^2$ and $\mathbf{e}^3$. As such we may obtain the following condition:

$$\begin{aligned} T^{\mu\nu} e^\alpha{}_\nu &= T^{\mu\alpha} \\ &= \mathcal{C}_\alpha S^\mu , \end{aligned}$$

where $\mathcal{C}_\alpha = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3)$ are constants. Explicitly:

$$\begin{aligned} T^{\mu 0} &= \mathcal{C}_0 S^\mu , \\ T^{\mu 1} &= \mathcal{C}_1 S^\mu , \\ T^{\mu 2} &= \mathcal{C}_2 S^\mu , \\ T^{\mu 3} &= \mathcal{C}_3 S^\mu . \end{aligned}$$

Thus the columns must be proportional to each other.

Exercise 2

Consider the following equation:

$$T^{\mu\nu} = U^{\mu} + V^{\nu} \ .$$

Is $\boldsymbol{T}$ a generic tensor?

Solution 2

$\boldsymbol{T}$ is not a generic tensor. If $\boldsymbol{T}$ were a tensor then $T^{\mu\nu} A_{\mu} B_{\nu}$ would have to be a scalar. Instead, one obtains

$$\begin{aligned} T^{\mu\nu} A_{\mu} B_{\nu} &= \left(U^{\mu} + V^{\nu}\right) A_{\mu} B_{\nu} \\ &= \left(U^{\mu} A_{\mu}\right) B_{\nu} + \left(V^{\nu} B_{\nu}\right) A_{\mu} \\ &= \alpha B_{\nu} + \beta A_{\mu} \ , \end{aligned}$$

where $\alpha \equiv U^{\mu} A_{\mu}$ and $\beta \equiv V^{\nu} B_{\nu}$ are both scalars. It immediately follows that $\alpha B_{\nu} + \beta A_{\mu}$ is not a scalar and therefore $\boldsymbol{T}$ is not a generic tensor.

Exercise 3

Consider $\boldsymbol{F}$ as a tensor of rank 2 with covariant components $F_{\mu\nu}$ and that is also antisymmetric in one co-ordinate system, i.e. $F_{\mu\nu} = -F_{\nu\mu}$.

- Show that $F_{\mu\nu}$ is antisymmetric in all co-ordinate systems.
- Does the antisymmetry in the covariant indices also apply to the contravariant indices?
- If so, show that $F^{\mu\nu}$ is antisymmetric in all co-ordinate systems.

Solution 3

First consider the transformation of $F_{\mu\nu}$ into another co-ordinate system:

$$\begin{aligned} F_{\mu'\nu'} &= \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\mu\nu} \\ &= -\Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\nu\mu} \\ &= -\Lambda^{\nu}{}_{\mu'} \Lambda^{\mu}{}_{\nu'} F_{\mu\nu} \\ &= -F_{\nu'\mu'} \ . \end{aligned}$$

It immediately follows that $F_{\mu\nu}$ is symmetric in all co-ordinate systems. The antisymmetry in covariant indices indeed also applies to the contravariant indices since $\boldsymbol{F}$ is a tensor. This can be shown by considering the following:

$$\begin{aligned} F^{\mu\nu} &= g^{\mu\mu'} g^{\nu\nu'} F_{\mu'\nu'} \\ &= -g^{\mu\mu'} g^{\nu\nu'} F_{\nu'\mu'} \\ &= -g^{\mu\nu'} g^{\nu\mu'} F_{\mu'\nu'} \\ &= -F^{\nu\mu} \ , \end{aligned}$$

as required.

Exercise 4

For the first part of the question, consider the antisymmetric tensor $A_{\mu\nu}$ such that $A_{\mu\nu} = -A_{\nu\mu}$ and the symmetric tensor $B^{\mu\nu}$ such that $B^{\mu\nu} = B^{\nu\mu}$. Prove the following identities:

$$A_{\mu\nu}B^{\mu\nu} = 0 \,, \tag{1}$$
$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} \,, \tag{2}$$
$$V^{\mu\nu}B_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)B_{\mu\nu} \,. \tag{3}$$

Solution 4

For the first identity consider the following:

$$\begin{aligned} A_{\mu\nu}B^{\mu\nu} &= -A_{\nu\mu}B^{\mu\nu} \\ &= -A_{\mu\nu}B^{\mu\nu} \\ &= 0 \,, \end{aligned}$$

where we have used the antisymmetry of $A_{\mu\nu}$ in the first step and relabelling dummy indices and the symmetry of $B^{\mu\nu}$ in the second step, hence the required result. For the second and third parts, recall that a generic rank 2 tensor may be written in terms of a symmetric and antisymmetric component as follows:

$$\begin{aligned} V^{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right) + \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right) \\ &= V^{(\mu\nu)} + V^{[\mu\nu]} \,. \end{aligned}$$

Now consider the action of the antisymmetric tensor $A_{\mu\nu}$ on the symmetric part of $V^{\mu\nu}$, i.e. $V^{(\mu\nu)}$:

$$\begin{aligned} V^{(\mu\nu)}A_{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} + V^{\nu\mu}A_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} + V^{\mu\nu}A_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} - V^{\mu\nu}A_{\mu\nu}\right) \\ &= 0 \,, \end{aligned}$$

where in the first step we have relabelled dummy indices in the second term, and in the second step we have used the antisymmetry of $A_{\mu\nu}$. In a similar fashion we may also consider the action of the symmetric tensor $B_{\mu\nu}$ on the antisymmetric part of $V^{\mu\nu}$, i.e. $V^{[\mu\nu]}$:

$$\begin{aligned} V^{[\mu\nu]}B_{\mu\nu} &= \frac{1}{2}\left(V^{\mu\nu}B_{\mu\nu} - V^{\nu\mu}B_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}B_{\mu\nu} - V^{\mu\nu}B_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}B_{\mu\nu} - V^{\mu\nu}B_{\mu\nu}\right) \\ &= 0 \,. \end{aligned}$$

We are now in a position to tackle the second and third identities. For the second identity, consider the following:

$$\begin{aligned} V^{\mu\nu}A_{\mu\nu} &= V^{(\mu\nu)}A_{\mu\nu} + V^{[\mu\nu]}A_{\mu\nu} \\ &= V^{[\mu\nu]}A_{\mu\nu} \\ &= \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} \ , \end{aligned}$$

as required.

Finally, we consider the third identity:

$$\begin{aligned} V^{\mu\nu}B_{\mu\nu} &= V^{(\mu\nu)}B_{\mu\nu} + V^{[\mu\nu]}B_{\mu\nu} \\ &= V^{(\mu\nu)}B_{\mu\nu} \\ &= \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)B_{\mu\nu} \ , \end{aligned}$$

as required.

General Relativity: Solutions to exercises in Lecture IV

January 22, 2018

Exercise 1

Using a co-ordinate system (t, r, θ, ϕ), consider the metric line element given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] , \tag{1}$$

where $\kappa = -1,\ 0,\ 1$.

- Show that a new co-ordinate system (t, χ, θ, ϕ) the line element (1) can be rewritten as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] . \tag{2}$$

- Find the form of the function $f(\chi)$ for $\kappa = -1$, 0 and 1.
- Discuss the properties of the metric in the case of $\kappa = 0$. [Hint: two metrics $\mathbf{g}$ and $\mathbf{g}'$ are conformally related if it is possible to express them as $\mathbf{g} = \Omega\,\mathbf{g}'$, where $\Omega \equiv \Omega(x^\mu)$ is a generic function and is referred to as the *conformal factor*].

Solution 1

From the invariance of the line element we may write

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \\
&= -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + f(\chi)^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \\
&= \mathrm{d}s'^2 .
\end{aligned}$$

Letting $\mathrm{d}t = \mathrm{d}\theta = \mathrm{d}\phi = 0$ we obtain the relation

$$\frac{\mathrm{d}r}{\sqrt{1-\kappa r^2}} = \mathrm{d}\chi . \tag{3}$$

This expression may be integrated directly to yield χ as a function of r, yielding:

$$\chi = \begin{cases} \operatorname{arcsinh} r + c \ , \\ r + c \ , \\ \arcsin r + c \ , \end{cases} \tag{4}$$

for $\kappa = -1$, 0 and 1 respectively, and where c is a constant of integration. Note the inverse hyperbolic function identity $\operatorname{arcsinh} r = \ln |r + \sqrt{1+r^2}|$, which is also a solution for $\kappa = -1$. Since $f(\chi) = r$ we obtain the result

$$f(\chi) = \begin{cases} \sinh \chi \ , \\ \chi \ , \\ \sin \chi \ , \end{cases} \tag{5}$$

for $\kappa = -1$, 0 and 1 respectively, and where we have assumed $c = 0$.

For the final part of this question, setting $\kappa = 0$ yields the line element as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \left[\mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \ . \tag{6}$$

By factoring out the expansion factor $a(t)$ we obtain

$$\mathrm{d}s^2 = a^2(t) \left[-\frac{\mathrm{d}t^2}{a^2(t)} + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \ . \tag{7}$$

Let us define the "conformal time" $\tilde{t}$, where $\mathrm{d}\tilde{t}^2 = \mathrm{d}t^2/a(t)^2$. We may now re-write the metric (7) as

$$\begin{aligned} \mathrm{d}s^2 &= a^2(t) \left[-\mathrm{d}\tilde{t}^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \\ &= a^2(t) \, \mathrm{d}s^2_{\text{Minkowski}} \ , \end{aligned} \tag{8}$$

where the Minkowski line element is the line element for flat space. Thus in the case $\kappa = 0$ the metric is conformally flat. This metric is in general known as the (Friedmann-Lemaitre) Robertson-Walker (FL)RW metric and is widely used in cosmology to describe an expanding universe.

Exercise 2

Using a co-ordinate system $(\eta, \chi, \theta, \phi)$, consider the metric line element given by

$$\mathrm{d}s^2 = \Omega^2 \left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \ . \tag{9}$$

Consider now a new co-ordinate system $(\tau, \rho, \theta, \phi)$ where

$$\tau = \frac{2\sin\eta}{\cos\chi + \cos\eta} \tag{10}$$

$$\rho = \frac{2\sin\chi}{\cos\chi + \cos\eta} \ , \tag{11}$$

and find the expression of the metric (9) in this new co-ordinate system. Discuss the properties of this new metric.

Solution 2

To calculate the expression for the new metric we must use the following co-ordinate transformation:

$$g^{\alpha'\beta'} = \Lambda^{\alpha'}_{\ \alpha}\Lambda^{\beta'}_{\ \beta}\, g^{\alpha\beta} \ . \tag{12}$$

We must calculate the contravariant metric components since we only have (τ, ρ) in terms of (η, χ) and not the inverse relationship. It is much simpler to calculate the contravaraint metric components and then calculate the matrix inverse of the contravariant metric than to define the inverse transformation. Since our metric is diagonal we may exploit the fact that $g^{\alpha\beta} = 1/g_{\alpha\beta}$. Evaluating the non-zero components of the transformation matrix, we obtain:

$$\begin{aligned} \Lambda^{0'}_{\ 0} &= \frac{\partial\tau}{\partial\eta} \\ &= \frac{2(1+\cos\eta\cos\chi)}{(\cos\chi+\cos\eta)^2} \\ &= \frac{\partial\rho}{\partial\chi} \\ &= \Lambda^{1'}_{\ 1} \ , \end{aligned} \tag{13}$$

$$\begin{aligned} \Lambda^{1'}_{\ 0} &= \frac{\partial\rho}{\partial\eta} \\ &= \frac{2\sin\chi\sin\eta}{(\cos\chi+\cos\eta)^2} \\ &= \frac{\partial\tau}{\partial\chi} \\ &= \Lambda^{0'}_{\ 1} \ , \end{aligned} \tag{14}$$

and

$$\Lambda^{2'}_{\ 2} = \Lambda^{3'}_{\ 3} = 1 \ . \tag{15}$$

Note that the transformation matrix is diagonal. The contravariant metric may now be calculated coefficient by coefficient, yielding

$$\begin{aligned} g^{0'0'} &= \left(\Lambda^{0'}_{\ 0}\right)^2 g^{00} + \left(\Lambda^{0'}_{\ 1}\right)^2 g^{11} \\ &= -\frac{1}{\Omega^2}\left[\left(\Lambda^{0'}_{\ 0}\right)^2 - \left(\Lambda^{0'}_{\ 1}\right)^2\right] \\ &= -\frac{1}{\Omega^2}\frac{4}{(\cos\chi+\cos\eta)^2} \ , \end{aligned} \tag{16}$$

$$\begin{aligned} g^{1'1'} &= \left(\Lambda^{1'}_{\ 0}\right)^2 g^{00} + \left(\Lambda^{1'}_{\ 1}\right)^2 g^{11} \\ &= \frac{1}{\Omega^2}\left[\left(\Lambda^{1'}_{\ 1}\right)^2 - \left(\Lambda^{1'}_{\ 0}\right)^2\right] \\ &= \frac{1}{\Omega^2}\left[\left(\Lambda^{0'}_{\ 0}\right)^2 - \left(\Lambda^{0'}_{\ 1}\right)^2\right] \\ &= -g^{0'0'} \ , \end{aligned} \tag{17}$$

$$
\begin{aligned}
g^{2'2'} &= \frac{1}{\Omega^2 \sin^2 \chi} \\
&= g^{22} \\
&= \frac{4}{\Omega^2 \left(\cos\chi + \cos\eta\right)^2 \rho^2} \\
&= -\frac{1}{\rho^2} g^{0'0'} \ ,
\end{aligned} \tag{18}
$$

and

$$
\begin{aligned}
g^{3'3'} &= \frac{1}{\Omega^2 \sin^2\theta \sin^2 \chi} \\
&= g^{33} \\
&= \frac{1}{\sin^2\theta} g^{2'2'} \\
&= -\frac{1}{\rho^2 \sin^2\theta} g^{0'0'} \ .
\end{aligned} \tag{19}
$$

Thus we may write the contravariant metric components in the new co-ordinate system as

$$
g^{\alpha'\beta'} = \frac{1}{\Omega^2} \frac{4}{\left(\cos\chi + \cos\eta\right)^2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \left(\rho^2\right)^{-1} & 0 \\ 0 & 0 & 0 & \left(\rho^2 \sin^2\theta\right)^{-1} \end{pmatrix} \ . \tag{20}
$$

Let us now define a new conformal factor

$$
\widetilde{\Omega}^2 \equiv \frac{\Omega^2 \left(\cos\chi + \cos\eta\right)^2}{4} \ , \tag{21}
$$

which immediately enables us to write the covariant components of our metric tensor in the new co-ordinate system as

$$
g_{\alpha'\beta'} = \widetilde{\Omega}^2 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ 0 & 0 & 0 & \rho^2 \sin^2\theta \end{pmatrix} \ . \tag{22}
$$

We may now write our metric in the new co-ordinate system as follows

$$
\mathrm{d}s^2 = \widetilde{\Omega}^2 \left(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\theta^2 + \rho^2 \sin^2\theta \, \mathrm{d}\phi^2\right) \ . \tag{23}
$$

As can be seen in the above expression, the new metric is conformally flat.

Exercise 3

Given the four-vector $\mathbf{u}$ such that $u^\alpha u_\alpha = -1$ and the tensor $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$, prove the following identities

$$h_{\mu\nu} u^\mu = 0 \ , \qquad h^\mu{}_\nu h^\lambda{}_\mu = h^\lambda{}_\nu \ , \qquad h^\mu{}_\mu = 3 \ . \tag{24}$$

Solution 3

For the first identity, consider the following

$$\begin{aligned} h_{\mu\nu} u^\mu &= g_{\mu\nu} u^\mu + u_\mu u_\nu u^\mu \\ &= u_\nu + u_\nu \left(u_\mu u^\mu \right) \\ &= u_\nu - u_\nu \\ &= 0 \ . \end{aligned} \tag{25}$$

For the second identity, we must first derive an expression for $h^\mu{}_\nu$ as follows

$$\begin{aligned} h^\mu{}_\nu &= g^{\mu\alpha} h_{\alpha\nu} \\ &= \delta^\mu{}_\nu + u^\mu u_\nu \ . \end{aligned} \tag{26}$$

Using this we may write the following

$$\begin{aligned} h^\mu{}_\nu h^\lambda{}_\mu &= \left(\delta^\mu{}_\nu + u^\mu u_\nu \right) \left(\delta^\lambda{}_\mu + u^\lambda u_\mu \right) \\ &= \delta^\mu{}_\nu \delta^\lambda{}_\mu + \delta^\mu{}_\nu u^\lambda u_\mu + \delta^\lambda{}_\mu u^\mu u_\nu + u^\mu u_\nu u^\lambda u_\mu \\ &= \delta^\lambda_\nu + u^\lambda u_\nu + u^\lambda u_\nu + u^\lambda u_\nu \left(u^\mu u_\mu \right) \\ &= \delta^\lambda_\nu + u^\lambda u_\nu \\ &= h^\lambda{}_\nu \ . \end{aligned} \tag{27}$$

For the third and final identity, consider the following

$$\begin{aligned} h^\mu{}_\mu &= g^{\mu\nu} h_{\nu\mu} \\ &= g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu} u_\mu u_\nu \\ &= \delta^\mu{}_\mu + u^\mu u_\mu \\ &= 4 - 1 \\ &= 3 \ . \end{aligned} \tag{28}$$

Note: the tensor $h_{\mu\nu}$ defines a projection onto a hypersurface orthogonal to u^μ (i.e. $h_{\mu\nu} u^\mu u^\nu = 0$). For any non-null vector u^μ (i.e. $u^\mu u_\mu \neq 0$), one may define the projection operator orthogonal to u^μ as

$$\begin{aligned} \mathrm{P}_{\underline{u}} &\equiv h_{\mu\nu} \\ &= g_{\mu\nu} - \frac{u_\mu u_\nu}{u_\mu u^\mu} \ . \end{aligned} \tag{29}$$

Exercise 4

Consider the following antisymmetric tensor

$$F_{\alpha\beta} = -2E_{[\alpha}u_{\beta]} + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta \ . \tag{30}$$

Express the vectors **E** and **H** in terms of the tensor **F**. [Hint: contract $F_{\alpha\beta}$ with u^β & $\epsilon^{\alpha\beta\gamma\delta}$ respectively.]

Solution 4

First, let us write the expression for the antisymmetric part of $E_\alpha u_\beta$ out in full, which reads as

$$E_{[\alpha}u_{\beta]} = \frac{1}{2}\left(E_\alpha u_\beta - E_\beta u_\alpha\right) \ . \tag{31}$$

We may then substitute this expression into equation (30), yielding

$$F_{\alpha\beta} = -E_\alpha u_\beta + E_\beta u_\alpha + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta \ . \tag{32}$$

Let us work with the above expression for the remainder of the question. Contracting $F_{\alpha\beta}$ with u^β yields

$$\begin{aligned} F_{\alpha\beta}u^\beta &= -E_\alpha u_\beta u^\beta + E_\beta u_\alpha u^\beta + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta \\ &= E_\alpha + E_\beta u_\alpha u^\beta + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta \ . \end{aligned} \tag{33}$$

Before proceeding further, let us turn our attention to the last term in equation (33). The Levi-Civita tensor may be re-written in a fully contravariant form as

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} = g_{\alpha\mu} \, g_{\beta\nu} \, \epsilon^{\mu\nu\gamma\delta} \ , \tag{34}$$

which simplifies the third term in equation (33) as follows

$$\begin{aligned} \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma \, u_\delta u^\beta &= g_{\alpha\mu} \, g_{\beta\nu} \, \epsilon^{\mu\nu\gamma\delta} u_\delta u^\beta \, H_\gamma \\ &= g_{\alpha\mu} \, \epsilon^{\mu\nu\gamma\delta} u_\delta u_\nu \, H_\gamma \qquad \text{(lower index with } g_{\beta\nu}\text{)} \\ &= g_{\alpha\mu} \, \epsilon^{\mu\delta\gamma\nu} u_\nu u_\delta \, H_\gamma \qquad (\delta \leftrightarrow \nu \text{ as dummy indices}) \\ &= -g_{\alpha\mu} \, \epsilon^{\mu\nu\gamma\delta} u_\nu u_\delta \, H_\gamma \qquad (\text{permute } \delta \leftrightarrow \nu \text{ in } \epsilon^{\mu\delta\gamma\nu}) \\ &= -g_{\alpha\mu} \, \epsilon^{\mu\nu\gamma\delta} u_\delta u_\nu \, H_\gamma \qquad \text{(compare with second line)} \\ &= 0 \ . \end{aligned} \tag{35}$$

We thus obtain

$$\begin{aligned} F_{\alpha\beta}u^\beta &= E_\alpha + E_\beta u_\alpha u^\beta \\ &= h^\beta{}_\alpha \, E_\beta \ , \end{aligned} \tag{36}$$

as required. This may also be written as

$$F_{\alpha\beta}u^\beta = h_{\alpha\beta} \, E^\beta \ . \tag{37}$$

For the second part of the question, first recall the definition of the dual of a tensor

$$F^{*\gamma\delta} = \frac{1}{2} F_{\alpha\beta}\, \epsilon^{\alpha\beta\gamma\delta} \ . \tag{38}$$

Contracting $F_{\alpha\beta}$ with $\epsilon^{\alpha\beta\gamma\delta}$ yields

$$\begin{aligned} F_{\alpha\beta}\epsilon^{\alpha\beta\gamma\delta} &= 2F^{*\gamma\delta} \\ &= -E_\alpha u_\beta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha\, \epsilon^{\alpha\beta\gamma\delta} + \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma\, u_\delta\, \epsilon^{\alpha\beta\gamma\delta} \ . \end{aligned} \tag{39}$$

Let us attack the third term in the above expression by employing the identity we derived in equation (34) as

$$\epsilon_{\alpha\beta}{}^{\gamma\delta} = g^{\mu\gamma} g^{\nu\delta} \epsilon_{\alpha\beta\mu\nu} \ . \tag{40}$$

With this expression we may re-write the third term as

$$\begin{aligned} \epsilon_{\alpha\beta}{}^{\gamma\delta} H_\gamma\, u_\delta\, \epsilon^{\alpha\beta\gamma\delta} &= g^{\mu\gamma} g^{\nu\delta}\, H_\gamma\, u_\delta\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} \\ &= H^\mu\, u^\nu\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} \ . \end{aligned} \tag{41}$$

We may then expand the contraction over the Levi-Civita tensors as

$$\begin{aligned} \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} &= -2!\, \delta^{\gamma\delta}_{\mu\nu} \\ &= -2 \begin{vmatrix} \delta^\gamma_\mu & \delta^\gamma_\nu \\ \delta^\delta_\mu & \delta^\delta_\nu \end{vmatrix} \\ &= 2\left(\delta^\gamma_\nu\, \delta^\delta_\mu - \delta^\gamma_\mu\, \delta^\delta_\nu\right) \ , \end{aligned} \tag{42}$$

from which we may immediately simplify equation (41), yielding

$$\begin{aligned} H^\mu\, u^\nu\, \epsilon_{\alpha\beta\mu\nu}\epsilon^{\alpha\beta\gamma\delta} &= 2H^\mu\, u^\nu\, \left(\delta^\gamma_\nu\, \delta^\delta_\mu - \delta^\gamma_\mu\, \delta^\delta_\nu\right) \\ &= 2\left(H^\delta u^\gamma - H^\gamma u^\delta\right) \end{aligned} \tag{43}$$

We may now re-write equation (39) as

$$2F^{*\gamma\delta} = -E_\alpha u_\beta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha\, \epsilon^{\alpha\beta\gamma\delta} + 2\left(H^\delta u^\gamma - H^\gamma u^\delta\right) \ . \tag{44}$$

Recall from equation (35) the vanishing of the contraction of the Levi-Civita tensor over two indices with two 4-vectors. This suggests to us that contracting equation (44) with u_δ will allow us to eliminate the first two terms in (44). With this knowledge in mind, contracting with u_δ yields

$$\begin{aligned} 2F^{*\gamma\delta} u_\delta &= -E_\alpha u_\beta u_\delta\, \epsilon^{\alpha\beta\gamma\delta} + E_\beta u_\alpha u_\delta\, \epsilon^{\alpha\beta\gamma\delta} + 2\left(H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta\right) \\ &= 2\left(H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta\right) \ , \end{aligned} \tag{45}$$

from which it immediately follows that

$$\begin{aligned} F^{*\gamma\delta} u_\delta &= H^\delta u^\gamma u_\delta - H^\gamma u^\delta u_\delta \\ &= H^\gamma + H^\delta u^\gamma u_\delta \ . \end{aligned} \tag{46}$$

As before, the above expressions may be written more succinctly in terms of the projection tensor as

$$\begin{aligned} F^{*\gamma\delta}u_\delta &= h^\gamma{}_\delta H^\delta \\ &= h^{\gamma\delta}H_\delta \; . \end{aligned} \tag{47}$$

For a physical interpretation consider an orthonormal comoving frame with $u^\mu = (1,0,0,0)$ and $u_\mu = (-1,0,0,0)$, i.e. $u^\mu u_\mu = -1$. In this frame

$$\begin{aligned} F_{\alpha\beta}u^\beta &= F_{\alpha 0} \\ &= E_\alpha + E_0 u_\alpha \; . \end{aligned} \tag{48}$$

When $\alpha = 0$

$$\begin{aligned} F_{00} &= E_0 - E_0 \\ &= 0 \; . \end{aligned} \tag{49}$$

Additionally

$$\begin{aligned} F_{i0} &= E_i + E_0 u_i \\ &= E_i \; , \end{aligned} \tag{50}$$

where $i = 1,2,3$. If $F_{\alpha\beta}$ is the electromagnetic field tensor then E_i is the 3-vector of the electric field. Next consider the dual tensor

$$\begin{aligned} F^{*\gamma\delta}u_\delta &= -F^{*\gamma 0} \\ &= H^\gamma - H^0 u^\gamma \; . \end{aligned} \tag{51}$$

When $\gamma = 0$ then

$$\begin{aligned} F^{*00} &= -\left(H^0 - H^0\right) \\ &= 0 \; . \end{aligned} \tag{52}$$

Additionally

$$\begin{aligned} F^{*i0} &= -(H^i - H^0 u^i) \\ &= -H^i \; , \end{aligned} \tag{53}$$

where again $i = 1,2,3$. H^i can be interpreted as the 3-vector of the magnetic field.

General Relativity: Solutions to exercises in Lecture V

January 22, 2018

Exercise 1

Let **F** be a rank-2 antisymmetric tensor, **G** a rank-2 symmetric tensor and **X** and rank-3 antisymmetric tensor. Provide explicit expressions for the following tensors: $F_{\mu\nu}$, $F_{[\mu\nu]}$, $F_{(\mu\nu)}$, $G_{[\mu\nu]}$, $G_{(\mu\nu)}$, $X_{[\alpha\beta\gamma]}$, $X_{(\alpha\beta\gamma)}$, $X_{[\alpha\beta]\gamma}$, $X_{(\alpha\beta)\gamma}$, $X_{[\alpha\beta](\gamma)}$ and $X_{(\alpha\beta)[\gamma]}$.

Solution 1

- $F_{\mu\nu} = -F_{\nu\mu}$
- $F_{[\mu\nu]} = \frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right) = \frac{1}{2}\left(F_{\mu\nu} + F_{\mu\nu}\right) = F_{\mu\nu}$
- $F_{(\mu\nu)} = \frac{1}{2}\left(F_{\mu\nu} + F_{\nu\mu}\right) = \frac{1}{2}\left(F_{\mu\nu} - F_{\mu\nu}\right) = 0$
- $G_{[\mu\nu]} = 0$ (the antisymmetric part of a totally symmetric tensor must be zero)
- $G_{(\mu\nu)} = G_{\mu\nu}$
- $X_{[\alpha\beta\gamma]} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} - X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} - X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} - X_{\gamma\beta\alpha}\right) = \frac{1}{6}\left(2X_{\alpha\beta\gamma} + 2X_{\gamma\alpha\beta} + 2X_{\beta\gamma\alpha}\right)$

  $= \frac{1}{3}\left(X_{\alpha\beta\gamma} + X_{\gamma\alpha\beta} + X_{\beta\gamma\alpha}\right)$
- $X_{(\alpha\beta\gamma)} = \frac{1}{3!}\left(X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma} + X_{\gamma\alpha\beta} + X_{\alpha\gamma\beta} + X_{\beta\gamma\alpha} + X_{\gamma\beta\alpha}\right)$

  $= \frac{1}{3!}\left(X_{\alpha\beta\gamma} - X_{\alpha\beta\gamma} + X_{\gamma\alpha\beta} - X_{\gamma\alpha\beta} + X_{\beta\gamma\alpha} - X_{\beta\gamma\alpha}\right) = 0$
- $X_{[\alpha\beta]\gamma} = \frac{1}{2}\left(X_{\alpha\beta\gamma} - X_{\beta\alpha\gamma}\right) = X_{\alpha\beta\gamma}$
- $X_{(\alpha\beta)\gamma} = \frac{1}{2}\left(X_{\alpha\beta\gamma} + X_{\beta\alpha\gamma}\right) = 0$
- $X_{[\alpha\beta](\gamma)} = X_{[\alpha\beta]\gamma} = X_{\alpha\beta\gamma}$
- $X_{(\alpha\beta)[\gamma]} = X_{(\alpha\beta)\gamma} = 0$

Exercise 2

Prove the following identities:

- $X_{((\alpha_1 \ \alpha_2 \ldots \alpha_n))} = X_{(\alpha_1 \ \alpha_2 \ldots \alpha_n)}$
- $X_{[[\alpha_1 \ \alpha_2 \ldots \alpha_n]]} = X_{[\alpha_1 \ \alpha_2 \ldots \alpha_n]}$
- $X_{(\alpha_1 \ldots [\alpha_l \ \alpha_m] \ldots \alpha_n)} = 0$
- $X_{[\alpha_1 \ldots [\alpha_l \ \alpha_m] \ldots \alpha_n]} = X_{[\alpha_1 \ldots \alpha_l \ \alpha_m \ldots \alpha_n]}$

Solution 2

- If $Y_{\alpha_1 \alpha_2 \ldots \alpha_n}$ is a totally symmetric tensor then we may write

  $$Y_{\alpha_{\pi_1} \alpha_{\pi_2} \ldots \alpha_{\pi_n}} = Y_{\alpha_1 \alpha_2 \ldots \alpha_n} \ , \tag{1}$$

  where π_i denotes permutation over the index i. We may thus write the symmetric part of $\mathbf{Y}$ as

  $$\begin{aligned} Y_{(\alpha_1 \alpha_2 \ldots \alpha_n)} &= \frac{1}{n!} \sum Y_{\alpha_{\pi_1} \alpha_{\pi_2} \ldots \alpha_{\pi_n}} \\ &= Y_{\alpha_1 \alpha_2 \ldots \alpha_n} \ . \end{aligned} \tag{2}$$

  Now, letting $X_{(\alpha_1 \alpha_2 \ldots \alpha_n)} = Y_{\alpha_1 \alpha_2 \ldots \alpha_n}$ we may write

  $$\begin{aligned} Y_{(\alpha_1 \alpha_2 \ldots \alpha_n)} &= X_{((\alpha_1 \alpha_2 \ldots \alpha_n))} \\ &= X_{(\alpha_1 \alpha_2 \ldots \alpha_n)} \ , \end{aligned} \tag{3}$$

  as required.

- Similarly to the previous question, if $Y_{\alpha_1 \alpha_2 \ldots \alpha_n}$ is a totally antisymmetric tensor then we may write

  $$(-1)^{\pi} Y_{\alpha_{\pi_1} \alpha_{\pi_2} \ldots \alpha_{\pi_n}} = Y_{\alpha_1 \alpha_2 \ldots \alpha_n} \ , \tag{4}$$

  We may thus write the antisymmetric part of $\mathbf{Y}$ as

  $$\begin{aligned} Y_{[\alpha_1 \alpha_2 \ldots \alpha_n]} &= \frac{1}{n!} \sum (-1)^{\pi} \, Y_{\alpha_{\pi_1} \alpha_{\pi_2} \ldots \alpha_{\pi_n}} \\ &= Y_{\alpha_1 \alpha_2 \ldots \alpha_n} \ . \end{aligned} \tag{5}$$

  Now, letting $X_{[\alpha_1 \alpha_2 \ldots \alpha_n]} = Y_{\alpha_1 \alpha_2 \ldots \alpha_n}$ we may write

  $$\begin{aligned} Y_{[\alpha_1 \alpha_2 \ldots \alpha_n]} &= X_{[[\alpha_1 \alpha_2 \ldots \alpha_n]]} \\ &= X_{[\alpha_1 \alpha_2 \ldots \alpha_n]} \ , \end{aligned} \tag{6}$$

  as required.

- By symmetry (outer round brackets) we have

$$X_{(\alpha_1 \ldots [\alpha_l \alpha_m] \ldots \alpha_n)} = X_{(\alpha_1 \ldots [\alpha_m \alpha_l] \ldots \alpha_n)} \ , \tag{7}$$

but by antisymmetry (inner square brackets) we have

$$X_{(\alpha_1 \ldots [\alpha_l \alpha_m] \ldots \alpha_n)} = -X_{(\alpha_1 \ldots [\alpha_m \alpha_l] \ldots \alpha_n)} \ , \tag{8}$$

and thus we may conclude that

$$X_{(\alpha_1 \ldots [\alpha_l \alpha_m] \ldots \alpha_n)} = 0 \ , \tag{9}$$

as required.

- First consider

$$X_{\alpha_1 \ldots [\alpha_l \alpha_m] \ldots \alpha_n} = \frac{1}{2} \left(X_{\alpha_1 \ldots \alpha_l \alpha_m \ldots \alpha_n} - X_{\alpha_1 \ldots \alpha_m \alpha_l \ldots \alpha_n} \right) \ . \tag{10}$$

Now take the full antisymmetric part of this

$$\begin{aligned} X_{[\alpha_1 \ldots [\alpha_l \alpha_m] \ldots \alpha_n]} &= \frac{1}{2n!} \sum (-1)^\pi \left(X_{\alpha_{\pi_1} \ldots \alpha_{\pi_l} \alpha_{\pi_m} \ldots \alpha_{\pi_n}} - X_{\alpha_{\pi_1} \ldots \alpha_{\pi_m} \alpha_{\pi_l} \ldots \alpha_{\pi_n}} \right) \\ &= \frac{1}{n!} \sum (-1)^\pi X_{\alpha_{\pi_1} \ldots \alpha_{\pi_l} \alpha_{\pi_m} \ldots \alpha_{\pi_n}} \\ &= X_{[\alpha_1 \ldots \alpha_l \alpha_m \ldots \alpha_n]} \ , \end{aligned} \tag{11}$$

where we have used the fact that $X_{\alpha_{\pi_1} \ldots \alpha_{\pi_m} \alpha_{\pi_l} \ldots \alpha_{\pi_n}} = -X_{\alpha_{\pi_1} \ldots \alpha_{\pi_l} \alpha_{\pi_m} \ldots \alpha_{\pi_n}}$, as required.

Exercise 3

Let $\mathbf{F}$ be a rank-2 antisymmetric tensor with components $F^{\mu\nu}$. From $\mathbf{F}$ construct another rank-2 tensor antisymmetric tensor ${}^*\mathbf{F}$ such that

$${}^*\mathbf{F} := \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \, \mathbf{e}_\mu \otimes \mathbf{e}_\nu \ . \tag{12}$$

The tensor ${}^*\mathbf{F}$ is usually referred to as the *dual* of $\mathbf{F}$. Show that the following is true

$${}^*\left({}^*\mathbf{F}\right) = -\mathbf{F} \ . \tag{13}$$

Solution 3

We may write the dual of $\mathbf{F}$ in contravariant index form as

$${}^*F^{\mu\nu} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} \ . \tag{14}$$

Accordingly, the covariant form may be written, using the relation $F_{\mu\nu} = g_{\mu\gamma} g_{\nu\delta} F^{\gamma\delta}$, as

$$\begin{aligned} {}^*F_{\mu\nu} &= \frac{1}{2} g_{\mu\gamma} g_{\nu\delta} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} \\ &= \frac{1}{2} \epsilon^{\alpha\beta}{}_{\mu\nu} F_{\alpha\beta} \\ &= \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} \epsilon_{\gamma\delta\mu\nu} F_{\alpha\beta} \\ &= \frac{1}{2} \epsilon_{\gamma\delta\mu\nu} F^{\gamma\delta} \ . \end{aligned} \tag{15}$$

We may now write

$$
\begin{aligned}
{}^*\left({}^*F^{\mu\nu}\right) &= \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\left({}^*F_{\alpha\beta}\right) \\
&= \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\gamma\delta\alpha\beta}F^{\gamma\delta} \\
&= \frac{1}{4}\left(-2!\,\delta^{\mu\nu}_{\gamma\delta}\right)F^{\gamma\delta} \\
&= -\frac{1}{2}\left(\delta^{\mu}_{\gamma}\,\delta^{\nu}_{\delta} - \delta^{\mu}_{\delta}\,\delta^{\nu}_{\gamma}\right)F^{\gamma\delta} \\
&= -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \\
&= -F^{\mu\nu}\ . \qquad (16)
\end{aligned}
$$

Thus we obtain

$$
{}^*\left({}^*\mathbf{F}\right) = -\mathbf{F}\ , \qquad (17)
$$

as required.

Exercise 4

Let $\mathbf{V}$ be a rank-3 tensor with components $V^{\alpha\beta\gamma}$ and define

$$
\left({}^*V\right)^{\alpha\beta\gamma} := V_{\mu}\epsilon^{\mu\alpha\beta\gamma}\ . \qquad (18)
$$

Show that the following is true

$$
V^{\mu}V_{\mu} = -\frac{1}{3!}\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma}\ . \qquad (19)
$$

Solution 4

In addition to equation (18), for fully covariant $\mathbf{V}$ we may write

$$
\left({}^*V\right)_{\alpha\beta\gamma} = V^{\nu}\epsilon_{\nu\alpha\beta\gamma}\ . \qquad (20)
$$

From this we may immediately calculate the contraction as

$$
\begin{aligned}
\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma} &= V^{\nu}V_{\mu}\epsilon^{\mu\alpha\beta\gamma}\epsilon_{\nu\alpha\beta\gamma} \\
&= V^{\nu}V_{\mu}\left(-3!\,\delta^{\mu}_{\nu}\right) \\
&= -3!\,V^{\mu}V_{\mu}\ , \qquad (21)
\end{aligned}
$$

and hence we obtain

$$
V^{\mu}V_{\mu} = -\frac{1}{3!}\left({}^*V\right)^{\alpha\beta\gamma}\left({}^*V\right)_{\alpha\beta\gamma}\ , \qquad (22)
$$

as required.

General Relativity: Solutions to exercises in Lecture VI

January 29, 2018

Exercise 1

Define the antisymmetric tensor $\mathbf{F}$ as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Use the results from the previous exercises to show that

$$F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]} \ . \tag{1}$$

Show that such a definition implies that

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0 \ . \tag{2}$$

Solution 1

For the first part we may simply write

$$\begin{aligned} F_{\mu\nu} &= 2\left[\frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right] \\ &= 2\,\partial_{[\mu} A_{\nu]} \ . \end{aligned} \tag{3}$$

For the second part of the question we must write out each of the three terms explicitly. For the first term in equation (2) we obtain

$$\begin{aligned} F_{\alpha\beta,\nu} &= \partial_\nu\left(F_{\alpha\beta}\right) \\ &= \partial_\nu\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\ &= \partial_\nu\partial_\alpha A_\beta - \partial_\nu\partial_\beta A_\alpha \ . \end{aligned} \tag{4}$$

For the second term

$$\begin{aligned} F_{\beta\nu,\alpha} &= \partial_\alpha\left(F_{\beta\nu}\right) \\ &= \partial_\alpha\left(\partial_\beta A_\nu - \partial_\nu A_\beta\right) \\ &= \partial_\alpha\partial_\beta A_\nu - \partial_\alpha\partial_\nu A_\beta \ . \end{aligned} \tag{5}$$

Finally, for the third term

$$\begin{aligned} F_{\nu\alpha,\beta} &= \partial_\beta\left(F_{\nu\alpha}\right) \\ &= \partial_\beta\left(\partial_\nu A_\alpha - \partial_\alpha A_\nu\right) \\ &= \partial_\beta\partial_\nu A_\alpha - \partial_\beta\partial_\alpha A_\nu \ . \end{aligned} \tag{6}$$

Since $\partial_\alpha\partial_\beta\mathbf{F} = \partial_\beta\partial_\alpha\mathbf{F}$, summing equations (4)–(6) leads to cancellation of terms, giving the result in equation (2), as required.

Exercise 2

Consider a vector $\mathbf{V}$ with components V^μ relative to a co-ordinate basis, i.e.

$$\mathbf{V} = V^\mu \partial_\mu = V^\mu \mathbf{e}_\mu \ . \tag{7}$$

Define an object given by the partial derivative of the components of $\mathbf{V}$, i.e.

$$U_\nu{}^\mu := \partial_\nu V^\mu \ . \tag{8}$$

Show that $U_\nu{}^\mu$ is not a tensor. What are the implications of this result? What can be done to construct a tensor out of measuring the derivative of a tensor?

Solution 2

From equation (8) we may write

$$\begin{aligned} U_\nu &= \partial_\nu \mathbf{V} \\ &= \partial_\nu \left(V^\mu \mathbf{e}_\mu \right) \\ &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\mu \partial_\nu \mathbf{e}_\mu \ . \end{aligned} \tag{9}$$

For the second term in the above expression we may think of it as a vector written in terms of some basis vectors. Let us re-write this as $\partial_\nu \mathbf{e}_\mu = \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha$. We may now write equation (9) as

$$\begin{aligned} U_\nu &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\mu \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha \\ &= \partial_\nu V^\mu \mathbf{e}_\mu + V^\alpha \Gamma^\mu_{\alpha\nu} \mathbf{e}_\mu \quad (\alpha \leftrightarrow \mu \text{ in the second term}) \\ &= \left(\partial_\nu V^\mu + V^\alpha \Gamma^\mu_{\alpha\nu} \right) \mathbf{e}_\mu \\ &= \left(\nabla_\nu V^\mu \right) \mathbf{e}_\mu \ , \end{aligned} \tag{10}$$

and thus we may write

$$U_\nu{}^\mu = \nabla_\nu V^\mu \ , \tag{11}$$

where the ∇_ν we have introduced is defined as the *covariant derivative*.

Consider the term $\partial_\nu \mathbf{e}_\mu = \Gamma^\alpha_{\mu\nu} \mathbf{e}_\alpha$. In flat (Minkowski) spacetime, in Cartesian co-ordinates, $\partial_\nu \mathbf{e}_\mu$ must vanish as the $\mathbf{e}_\mu$ are all constant, and thus $\Gamma^\alpha_{\mu\nu}$ must also be zero. However, in the same Minkowski spacetime, transforming to (for example) spherical polar co-ordinates one would find the basis vector components are not constant and are in fact functionally dependent on r and θ. As such, $\partial_\nu \mathbf{e}_\mu$ would be non-zero in Minkowski spacetime. Since a tensor quantity is defined independently of any co-ordinate system, the quantity $U_\nu{}^\mu$ cannot be a tensor.

The partial derivative is not a good differential operator when spacetime is not Euclidean but by construction the covariant derivative does define the components of a tensor.

Exercise 3

Consider a line element in three-dimensional space

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2 \ , \tag{12}$$

with a co-ordinate basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$.

- Construct the corresponding orthonormal basis $\{\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\}$
- Compute the structure coefficients $C^{\theta}_{\hat{r}\hat{\theta}}$ and $C^{\theta}_{r\theta}$. What is the difference between the two?
- Compute the structure coefficients $C^{\theta}_{\hat{r}\hat{\phi}}$, $C^{\phi}_{\hat{r}\hat{\phi}}$, $C^{\theta}_{\hat{\theta}\hat{\phi}}$ and $C^{\phi}_{\hat{\theta}\hat{\phi}}$. Are there others that are non-zero?

Solution 3

Since our metric is diagonal we can immediately read of the orthonormal basis vector components as

$$\mathbf{e}_{\hat{r}} = \mathbf{e}_r \ , \qquad \mathbf{e}_{\hat{\theta}} = \frac{1}{r}\mathbf{e}_\theta \ , \qquad \mathbf{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\mathbf{e}_\phi \ , \tag{13}$$

from which it is straightforward to show that $\mathbf{e}_{\hat{r}} \cdot \mathbf{e}_{\hat{r}} = 1$, $\mathbf{e}_{\hat{\theta}} \cdot \mathbf{e}_{\hat{\theta}} = 1$ and $\mathbf{e}_{\hat{\phi}} \cdot \mathbf{e}_{\hat{\phi}} = 1$. To convince ourselves this is correct, consider the transformation between the co-ordinate basis and orthonormal basis

$$\mathrm{d}x^{\hat{i}} = \Lambda^{\hat{i}}{}_j \, \mathrm{d}x^j \ , \tag{14}$$

where

$$\Lambda^{\hat{i}}{}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix} \ . \tag{15}$$

Using the co-ordinate transformation we may show that

$$\begin{aligned} \mathrm{d}\hat{r} &= \mathrm{d}x^{\hat{1}} \\ &= \Lambda^{\hat{1}}{}_j \mathrm{d}x^j \\ &= \Lambda^{\hat{1}}{}_1 \mathrm{d}x^1 \\ &= \mathrm{d}r \ . \end{aligned} \tag{16}$$

Similarly, one may show that

$$\mathrm{d}\hat{\theta} = r\,\mathrm{d}\theta \ , \tag{17}$$
$$\mathrm{d}\hat{\phi} = r\sin\theta\,\mathrm{d}\phi \ . \tag{18}$$

Now let us write the line element in terms of the orthonormal basis components and prove equivalence

$$\begin{aligned} \mathrm{d}s^2 &= g_{\hat{r}\hat{r}}\mathrm{d}\hat{r}^2 + g_{\hat{\theta}\hat{\theta}}\mathrm{d}\hat{\theta}^2 + g_{\hat{\phi}\hat{\phi}}\mathrm{d}\hat{\phi}^2 \\ &= \mathrm{d}\hat{r}^2 + \mathrm{d}\hat{\theta}^2 + \mathrm{d}\hat{\phi}^2 \\ &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \ , \end{aligned} \tag{19}$$

hence the orthonormal basis vector components are correct.

For the next part of the question recall the definition of the Lie brackets of any two basis vectors, which may be written in terms of the same basis as

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = C^{\gamma}_{\alpha\beta}\mathbf{e}_\gamma \ , \tag{20}$$

where the components $C^{\gamma}_{\alpha\beta}$ are termed the structure coefficients. By definition a set of basis vectors with all of its structure coefficients vanishing is a co-ordinate basis. We may now write

$$\begin{aligned} C^{\gamma}_{\alpha\beta} &= [\mathbf{e}_\alpha, \mathbf{e}_\beta]^\gamma \\ &= \mathbf{e}_\alpha{}^\nu \partial_\nu \, \mathbf{e}_\beta{}^\gamma - \mathbf{e}_\beta{}^\nu \partial_\nu \, \mathbf{e}_\alpha{}^\gamma \; . \end{aligned} \tag{21}$$

For the first structure coefficient, applying the above machinery we find

$$\begin{aligned} C^{\theta}_{\hat{r}\hat{\theta}} &= [\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}]^\theta \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\theta}}{}^\theta - \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{r}}{}^\theta \\ &= \mathbf{e}_{\hat{r}}{}^r \partial_r \left(\frac{1}{r}\right) - \mathbf{e}_{\hat{\theta}}{}^\theta \partial_\theta \, \cancel{\mathbf{e}_{\hat{r}}{}^\theta} \\ &= -\frac{1}{r^2} \; . \end{aligned} \tag{22}$$

As mentioned previously, $C^{\theta}_{r\theta} = 0$ since $\{\mathbf{e}_i\}$ is a co-ordinate basis. For the final four requested structure components we apply the same procedure for calculation. The results are as follows

$$\begin{aligned} C^{\theta}_{\hat{r}\hat{\phi}} &= \left[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\right]^\theta \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{\phi}}{}^\theta} - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{r}}{}^\theta} \\ &= 0 \; , \end{aligned} \tag{23}$$

$$\begin{aligned} C^{\phi}_{\hat{r}\hat{\phi}} &= \left[\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\phi}}\right]^\phi \\ &= \mathbf{e}_{\hat{r}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\phi}}{}^\phi - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{r}}{}^\phi \\ &= \mathbf{e}_{\hat{r}}{}^r \partial_r \, \mathbf{e}_{\hat{\phi}}{}^\phi - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{r}}{}^\phi} \\ &= \partial_r \left(\frac{1}{r\sin\theta}\right) \\ &= -\frac{1}{r^2 \sin\theta} \; , \end{aligned} \tag{24}$$

$$\begin{aligned} C^{\theta}_{\hat{\theta}\hat{\phi}} &= \left[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\right]^\theta \\ &= \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\phi}}{}^\theta - \mathbf{e}_{\hat{\phi}}{}^\nu \partial_\nu \, \mathbf{e}_{\hat{\theta}}{}^\theta \\ &= \mathbf{e}_{\hat{\theta}}{}^\nu \partial_\nu \, \cancel{\mathbf{e}_{\hat{\phi}}{}^\theta} - \mathbf{e}_{\hat{\phi}}{}^\phi \partial_\phi \, \mathbf{e}_{\hat{\theta}}{}^\theta \\ &= -\frac{1}{r\sin\theta} \partial_\phi \left(r\right) \\ &= 0 \; , \end{aligned} \tag{25}$$

and

$$
\begin{aligned}
C^{\phi}_{\hat{\theta}\hat{\phi}} &= \Big[\mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\phi}}\Big]^{\phi} \\
&= \mathbf{e}_{\hat{\theta}}{}^{\nu}\partial_{\nu}\,\mathbf{e}_{\hat{\phi}}{}^{\phi} - \mathbf{e}_{\hat{\phi}}{}^{\nu}\partial_{\nu}\,\mathbf{e}_{\hat{\theta}}{}^{\phi} \\
&= \mathbf{e}_{\hat{\theta}}{}^{\theta}\partial_{\theta}\,\mathbf{e}_{\hat{\phi}}{}^{\phi} - \mathbf{e}_{\hat{\phi}}{}^{\nu}\partial_{\nu}\,\cancel{\mathbf{e}_{\hat{\theta}}{}^{\phi}} \\
&= \frac{1}{r}\partial_{\theta}\left(\frac{1}{r\sin\theta}\right) \\
&= -\frac{\cos\theta}{r^2\sin^2\theta}\ .
\end{aligned}
\tag{26}
$$

There are no other non-zero structure coefficients.

General Relativity: Solutions to exercises in Lecture VII

January 29, 2018

Exercise 1

Show that if **g** is the metric tensor, then its covariant derivative is zero, i.e.

$$\nabla_\lambda g_{\mu\nu} = 0 \ . \tag{1}$$

Solution 1

By definition $\boldsymbol{\nabla} A_\mu$ is a vector. As such we may write

$$\nabla_\lambda A_\mu = g_{\mu\nu} \left(\nabla_\lambda A^\nu \right) \ . \tag{2}$$

We may also write

$$\begin{aligned} \nabla_\lambda A_\mu &= \nabla_\lambda \left(g_{\mu\nu} A^\nu \right) \\ &= \left(\nabla_\lambda g_{\mu\nu} \right) A^\nu + g_{\mu\nu} \left(\nabla_\lambda A^\nu \right) \ . \end{aligned} \tag{3}$$

Using equation (2) we may rewrite the above expression as

$$\nabla_\lambda A_\mu = \left(\nabla_\lambda g_{\mu\nu} \right) A^\nu + \nabla_\lambda A_\mu \ , \tag{4}$$

from which it immediately follows that

$$\nabla_\lambda g_{\mu\nu} = 0 \ , \tag{5}$$

as required.

Exercise 2

Using the results of exercise 1, drive the following definition of the Christoffel symbols

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(\partial_\gamma g_{\delta\beta} + \partial_\beta g_{\delta\gamma} - \partial_\delta g_{\beta\gamma} \right) \ . \tag{6}$$

Solution 2

Consider the following three expressions for the covariant derivative of the (covariant) metric tensor

$$\begin{aligned}
\nabla_\lambda g_{\mu\nu} &= g_{\mu\nu,\lambda} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha} \quad (=0)\ , &(7)\\
\nabla_\mu g_{\nu\lambda} &= g_{\nu\lambda,\mu} - \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} g_{\nu\alpha} \quad (=0)\ , &(8)\\
\nabla_\nu g_{\lambda\mu} &= g_{\lambda\mu,\nu} - \Gamma^\alpha_{\nu\lambda} g_{\alpha\mu} - \Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} \quad (=0)\ , &(9)
\end{aligned}$$

where we have (evenly) permuted the covariant indices, as well as having made use of the result of exercise 1, namely that $\nabla_\lambda g_{\mu\nu} = 0$. We have also written partial derivatives as subscript commas for the sake of brevity.

To prove the result, subtract the last two expressions from the first, i.e. (7) $-$[(8) + (9)]. This yields

$$g_{\mu\nu,\lambda} - g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu} - \boldsymbol{\Gamma^\alpha_{\lambda\mu} g_{\alpha\nu}} - \boldsymbol{\Gamma^\alpha_{\lambda\nu} g_{\mu\alpha}} + \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \boldsymbol{\Gamma^\alpha_{\mu\lambda} g_{\nu\alpha}} + \boldsymbol{\Gamma^\alpha_{\nu\lambda} g_{\alpha\mu}} + \Gamma^\alpha_{\nu\mu} g_{\lambda\alpha} = 0\ . \qquad (10)$$

By the torsion-free condition (i.e. $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$) and symmetry of the metric tensor (i.e. $g_{\mu\nu} = g_{\nu\mu}$) the red and blue terms cancel, yielding

$$g_{\mu\nu,\lambda} - g_{\nu\lambda,\mu} - g_{\lambda\mu,\nu} + 2\,\Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} = 0\ , \qquad (11)$$

which upon re-arranging gives

$$g_{\alpha\lambda}\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)\ . \qquad (12)$$

Multiplying both sides by $g^{\beta\lambda}$ gives

$$\delta^\beta_\alpha \Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\beta\lambda}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)\ , \qquad (13)$$

which immediately simplifies to

$$\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\beta\lambda}\left(g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}\right)\ . \qquad (14)$$

Finally, making the substitutions $\beta \to \alpha$, $\mu \to \beta$, $\nu \to \gamma$ and $\lambda \to \delta$ we obtain the result

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right)\ , \qquad (15)$$

as required.

Exercise 3

Prove the following identities:

$$\begin{aligned}
\partial_\gamma g_{\alpha\beta} &= \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}\ , &(16)\\
g_{\alpha\mu}\partial_\gamma g^{\mu\beta} &= -g^{\mu\beta}\partial_\gamma g_{\alpha\mu}\ , &(17)\\
\partial_\gamma g^{\alpha\beta} &= -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha}\ , &(18)\\
(\ln|g|)_{,\alpha} &= g^{\mu\nu} g_{\mu\nu,\alpha}\ , &(19)\\
\nabla_\mu A^\mu &= \frac{1}{|g|^{1/2}}\partial_\mu\left(|g|^{1/2}A^\mu\right) \quad \text{in a coordinate basis.} &(20)
\end{aligned}$$

Solution 3

- For the first part consider the action of the covariant derivative on $g_{\alpha\beta}$:

$$\nabla_\gamma g_{\alpha\beta} = g_{\alpha\beta,\gamma} - \Gamma^\lambda_{\gamma\alpha} g_{\lambda\beta} - \Gamma^\lambda_{\gamma\beta} g_{\alpha\lambda} = 0 \ . \tag{21}$$

  Rearranging yields

$$\begin{aligned} g_{\alpha\beta,\gamma} &= \Gamma^\lambda_{\gamma\alpha} g_{\lambda\beta} + \Gamma^\lambda_{\gamma\beta} g_{\alpha\lambda} \\ &= g_{\lambda\beta}\Gamma^\lambda_{\alpha\gamma} + g_{\alpha\lambda}\Gamma^\lambda_{\beta\gamma} \\ &= \Gamma_{\beta\alpha\gamma} + \Gamma_{\alpha\beta\gamma} \ , \end{aligned} \tag{22}$$

  as required.

- For the second part consider the following expression:

$$\left(g_{\alpha\mu} g^{\mu\beta}\right)_{,\gamma} = \left(\delta^\beta_\alpha\right)_{,\gamma} = 0 \ , \tag{23}$$

  which may also be expanded as

$$\begin{aligned} \left(g_{\alpha\mu} g^{\mu\beta}\right)_{,\gamma} &= g_{\alpha\mu,\gamma} g^{\mu\beta} + g_{\alpha\mu} g^{\mu\beta}{}_{,\gamma} \\ &= 0 \ . \end{aligned} \tag{24}$$

  Rearranging the above expression, and writing partial derivatives explicitly, we obtain

$$g_{\alpha\mu}\partial_\gamma g^{\mu\beta} = -g^{\mu\beta}\partial_\gamma g_{\alpha\mu} \ , \tag{25}$$

  as required.

- For the third part, let us consider the action of the covariant derivative on the contravariant metric tensor:

$$\nabla_\gamma g^{\alpha\beta} = g^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha_{\gamma\mu} g^{\mu\beta} + \Gamma^\beta_{\gamma\mu} g^{\alpha\mu} = 0 \ . \tag{26}$$

  Rearranging and making use of the symmetry conditions of the metric tensor and Christoffel symbol yields

$$\partial_\gamma g^{\alpha\beta} = -\Gamma^\alpha_{\mu\gamma} g^{\mu\beta} - \Gamma^\beta_{\mu\gamma} g^{\mu\alpha} \ , \tag{27}$$

  as required.

- For the fourth part consider the metric tensor $g_{\alpha\beta}$, which is a rank-2 tensor and specifically a matrix. For matrices one may consider the Jacobi matrix formula:

$$\frac{\partial}{\partial x^\alpha} \det\left[g_{\mu\nu}(x^\alpha)\right] = \mathrm{Tr}\left[\mathrm{adj}\left(g_{\alpha\beta}(x^\alpha)\right) \frac{\partial g_{\mu\nu}(x^\alpha)}{\partial x^\alpha}\right] \ , \tag{28}$$

  where the adjugate of a matrix may be written as

$$\begin{aligned} \mathrm{adj}\left(g_{\alpha\beta}\right) &= \det\left(g_{\alpha\beta}\right)\left(g_{\alpha\beta}\right)^{-1} \\ &= g^{\alpha\beta}\det\left(g_{\alpha\beta}\right) \\ &= g^{\alpha\beta}|g| \ , \end{aligned} \tag{29}$$

where we have omitted writing the dependence of the metric on co-ordinates for brevity, and written the determinant of the metric tensor as $|g|$.

We may now rewrite the Jacobi identity in equation (28) in logarithmic form as

$$\begin{aligned} \frac{\partial}{\partial x^\alpha}\left[\ln\det\left(g_{\alpha\beta}\right)\right] &= \frac{1}{\det\left(g_{\alpha\beta}\right)}\frac{\partial}{\partial x^\alpha}\left[\det\left(g_{\alpha\beta}\right)\right] \\ &= \frac{1}{|g|}\mathrm{Tr}\left[g^{\alpha\beta}|g|g_{\mu\nu,\alpha}\right] \\ &= \mathrm{Tr}\left[g^{\alpha\beta}g_{\mu\nu,\alpha}\right] \\ &= g^{\mu\nu}g_{\mu\nu,\alpha}\ . \end{aligned} \tag{30}$$

This may be written more succinctly as

$$\left(\ln|g|\right)_{,\alpha} = g^{\mu\nu}g_{\mu\nu,\alpha}\ , \tag{31}$$

as required.

- For the fifth and final part, consider the action of the covariant derivative on A^μ:

$$\nabla_\mu A^\mu = A^\mu{}_{,\mu} + \Gamma^\mu_{\mu\nu}A^\nu\ . \tag{32}$$

Using the definition of the covariant derivative derived in exercise 2, we may write $\Gamma^\mu_{\mu\nu}$ as:

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\delta}\left(g_{\delta\mu,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right)\ . \tag{33}$$

Whilst it is not immediately obvious, it can be shown that the last two terms in brackets in equation (33) vanish. Consider the following:

$$\begin{aligned} g^{\mu\delta}\left(g_{\nu\delta,\mu} - g_{\mu\nu,\delta}\right) &= g^{\mu\delta}\partial_\mu g_{\nu\delta} - g^{\mu\delta}\partial_\delta g_{\mu\nu} \\ &= \partial^\delta g_{\nu\delta} - \partial^\mu g_{\mu\nu} \\ &= \partial^\delta g_{\delta\nu} - \partial^\mu g_{\mu\nu} \quad \left(g_{\nu\delta} = g_{\delta\nu}\right) \\ &= \partial^\mu g_{\mu\nu} - \partial^\mu g_{\mu\nu} \quad (\delta \text{ is a dummy index}) \\ &= 0\ . \end{aligned} \tag{34}$$

Consequently we may rewrite equation (33) as

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2}g^{\mu\delta}g_{\delta\mu,\nu}\ . \tag{35}$$

In the fourth part of this exercise we showed that $\left(\ln|g|\right)_{,\nu} = g^{\mu\delta}g_{\mu\delta,\nu}$. Using this we may write

$$\begin{aligned} \Gamma^\mu_{\mu\nu} &= \frac{1}{2}g^{\mu\delta}g_{\delta\mu,\nu} \\ &= \frac{1}{2}\left(\ln|g|\right)_{,\nu} \\ &= \left(\ln|g|^{1/2}\right)_{,\nu} \\ &= \frac{\left(|g|^{1/2}\right)_{,\nu}}{|g|^{1/2}} \end{aligned} \tag{36}$$

Returning to equation (32) we may now re-write the expression as

$$\begin{aligned}
\nabla_\mu A^\mu &= A^\mu{}_{,\mu} + \frac{\left(|g|^{1/2}\right)_{,\nu}}{|g|^{1/2}} A^\nu \\
&= A^\mu{}_{,\mu} + \frac{\left(|g|^{1/2}\right)_{,\mu}}{|g|^{1/2}} A^\mu \quad \text{(relabel dummy index)} \\
&= \frac{1}{|g|^{1/2}} \left[|g|^{1/2} A^\mu{}_{,\mu} + \left(|g|^{1/2}\right)_{,\mu} A^\mu \right] \\
&= \frac{1}{|g|^{1/2}} \left(|g|^{1/2} A^\mu \right)_{,\mu} \\
&\equiv \frac{1}{|g|^{1/2}} \partial_\mu \left(|g|^{1/2} A^\mu \right) \ ,
\end{aligned} \tag{37}$$

as required.

Exercise 4

Optional: The covariant derivative of a contravariant vector U^μ is

$$\nabla_\nu U^\mu := \partial_\nu U^\mu + \Gamma^\mu_{\nu\lambda} U^\lambda \ . \tag{38}$$

Use this expression to obtain the covariant derivative of the covariant vector U_μ.

Solution 4

There are several ways one can go about proving this. Let us consider two such methods.

- **Method 1**

  Consider the following:

$$\begin{aligned}
\nabla_\nu \left(V^\mu U_\mu \right) &= V^\mu_{;\nu} U_\mu + V^\mu U_{\mu;\nu} \\
&= V^\mu_{,\nu} U_\mu + \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ ,
\end{aligned} \tag{39}$$

  where the subscript $_{;\nu}$ denotes the covariant differentiation with respect to x^ν and we have used the definition of $V^\mu_{;\nu}$. Since the quantity $V^\mu U_\mu$ is a scalar we may also write

$$\begin{aligned}
\nabla_\nu \left(V^\mu U_\mu \right) &= \partial_\nu \left(V^\mu U_\mu \right) \\
&= V^\mu_{,\nu} U_\mu + V^\mu U_{\mu,\nu} \ .
\end{aligned} \tag{40}$$

  Combining the above two equations yields

$$V^\mu_{,\nu} U_\mu + V^\mu U_{\mu,\nu} = V^\mu_{,\nu} U_\mu + \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ , \tag{41}$$

  which simplifies to

$$V^\mu U_{\mu,\nu} = \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu + V^\mu U_{\mu;\nu} \ , \tag{42}$$

from which we may obtain

$$V^\mu U_{\mu;\nu} = V^\mu U_{\mu,\nu} - \Gamma^\mu_{\nu\lambda} V^\lambda U_\mu \ . \tag{43}$$

Now let us set $V^\mu = \delta^\mu_\beta$, which gives

$$\delta^\mu_\beta U_{\mu;\nu} = \delta^\mu_\beta U_{\mu,\nu} - \Gamma^\mu_{\nu\lambda} \delta^\lambda_\beta U_\mu \ , \tag{44}$$

which simplifies to

$$U_{\beta;\nu} = U_{\beta,\nu} - \Gamma^\mu_{\nu\beta} U_\mu \ , \tag{45}$$

where upon setting $\mu \leftrightarrow \alpha$ and then $\beta \rightarrow \mu$ we obtain

$$U_{\mu;\nu} = U_{\mu,\nu} - \Gamma^\alpha_{\mu\nu} U_\alpha \ , \tag{46}$$

as required.

- **Method 2**

$$\begin{aligned} \nabla_\nu U_\mu &= \nabla_\nu \left(g_{\mu\alpha} U^\alpha \right) \\ &= \cancel{(\nabla_\nu g_{\mu\alpha})} + g_{\mu\alpha} \nabla_\nu U^\alpha \\ &= g_{\mu\alpha} \left(U^\alpha_{\ ,\nu} + \Gamma^\alpha_{\nu\lambda} U^\lambda \right) \\ &= g_{\mu\alpha} \left(U^\alpha_{\ ,\nu} \right) + g_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} U^\lambda \ . \end{aligned} \tag{47}$$

Now consider the expression

$$\begin{aligned} \left(g_{\mu\alpha} U^\alpha \right)_{,\nu} &= U_{\mu,\nu} \\ &= g_{\mu\alpha,\nu} U^\alpha + g_{\mu\alpha} \left(U^\alpha_{\ ,\nu} \right) \ , \end{aligned} \tag{48}$$

which upon rearrangement yields

$$g_{\mu\alpha} \left(U^\alpha_{\ ,\nu} \right) = U_{\mu,\nu} - g_{\mu\alpha,\nu} U^\alpha \ . \tag{49}$$

Substituting equation (49) into equation (47) yields

$$\nabla_\nu U_\mu = U_{\mu,\nu} - g_{\mu\alpha,\nu} U^\alpha + g_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} U^\lambda \ . \tag{50}$$

From exercise 3, part 1, recall the identity

$$g_{\mu\alpha,\nu} = g_{\lambda\alpha} \Gamma^\lambda_{\nu\mu} + g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} \ . \tag{51}$$

This enables us to rewrite the last two terms in equation (50) as:

$$\begin{aligned} -g_{\mu\alpha,\nu} U^\alpha + g_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} U^\lambda &= -g_{\lambda\alpha} \Gamma^\lambda_{\nu\mu} U^\alpha - g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} U^\alpha + g_{\mu\alpha} \Gamma^\alpha_{\nu\lambda} U^\lambda \\ &= -g_{\lambda\alpha} \Gamma^\lambda_{\nu\mu} - g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} U^\alpha + g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} U^\alpha \quad (\alpha \leftrightarrow \lambda \text{ in last term}) \\ &= -g_{\lambda\alpha} \Gamma^\lambda_{\nu\mu} u^\alpha \\ &= -\Gamma^\lambda_{\nu\mu} u_\lambda \ . \end{aligned} \tag{52}$$

We may now use the above expression to rewrite equation (50) as

$$\nabla_\nu U_\mu = U_{\mu,\nu} - \Gamma^\lambda_{\nu\mu} U_\lambda \ , \tag{53}$$

which may be rewritten as

$$\nabla_\nu U_\mu := \partial_\nu U_\mu - \Gamma^\lambda_{\nu\mu} U_\lambda \ , \tag{54}$$

as required.

General Relativity: Solutions to exercises in Lecture VIII

January 29, 2018

Exercise 1

Consider the metric describing, in polar co-ordinates (r, θ), a Euclidean space

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\theta^2 \ . \tag{1}$$

- Calculate the Christoffel symbols and geodesic curves associated with this space, which are given by the geodesic equation

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0 \ . \tag{2}$$

- Combine the two second-order differential equations describing the geodesic curves into a single first-order differential equation for $r = r(\theta)$.

- What is the differential equation for a straight line in this space?

Solution 1

- First let us consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \ , \tag{3}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \ , \tag{4}$$

$$g_{\mu\nu,r} = \begin{pmatrix} 0 & 0 \\ 0 & 2r \end{pmatrix} \ , \tag{5}$$

$$g_{\mu\nu,\theta} = 0 \ . \tag{6}$$

  Next, recall the definition of the Christoffel symbols:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}\right) \ . \tag{7}$$

Since α can only be r or θ and the metric is diagonal, we may proceed as follows:

$$
\begin{aligned}
\Gamma^r_{\beta\gamma} &= \frac{1}{2} g^{rr} \left(\cancel{g_{r\beta,\gamma}} + \cancel{g_{\gamma r,\beta}} - g_{\beta\gamma,r} \right) \\
&= -\frac{1}{2} g^{rr} g_{\beta\gamma,r} \ , \qquad (8) \\
\Gamma^\theta_{\beta\gamma} &= \frac{1}{2} g^{\theta\theta} \left(g_{\theta\beta,\gamma} + g_{\gamma\theta,\beta} - \cancel{g_{\beta\gamma,\theta}} \right) \\
&= \frac{1}{2} g^{\theta\theta} g_{\theta\beta,\gamma} \ . \qquad (9)
\end{aligned}
$$

It immediately follows that the only non-zero Christoffel symbols are given by:

$$
\Gamma^r_{\theta\theta} = -r \ , \qquad (10)
$$

$$
\Gamma^\theta_{r\theta} = \frac{1}{r} \ . \qquad (11)
$$

Substituting these expression into the geodesic equation of motion (2) we obtain:

$$
\ddot{r} = r\dot{\theta}^2 \ , \qquad (12)
$$

$$
\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} \ , \qquad (13)
$$

where an overdot denotes differentiation with respect to the affine parameter, λ.

- For the second part of the question we may rewrite equation (13) as:

$$
\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(r^2 \dot{\theta} \right) = 0 \ , \qquad (14)
$$

which may be integrated to yield

$$
\dot{\theta} = \frac{k}{r^2} \ , \qquad (15)
$$

where k is a constant of integration. Next, starting from the line element and dividing both sides by $\mathrm{d}s^2$ and taking s as affine we may write

$$
\dot{r}^2 + r^2 \dot{\theta}^2 = 1 \ . \qquad (16)
$$

Using the chain rule we may write equation (16) as:

$$
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \right)^2 + r^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \right)^2 = 1 \ . \qquad (17)
$$

Substituting equation (15) into equation (17) we obtain:

$$
\left[r'(\theta)^2 + r^2 \right] \frac{k^2}{r^4} = 1 \ , \qquad (18)
$$

where a primed quantity denotes differentiation with respect to θ. This may be simplified to yield

$$
r'(\theta) = \pm r \sqrt{\frac{r^2}{k^4} - 1} \ . \qquad (19)
$$

- For the final part of the question, let us integrate equation (19), which describes geodesics in this spacetime. Rearranging both sides of equation (19) gives:

$$\frac{\mathrm{d}r}{r\sqrt{\frac{r^2}{k^4}-1}} = \pm \mathrm{d}\theta \ . \tag{20}$$

Integrating both sides of the above equation then yields:

$$\arctan\left(\sqrt{\frac{r^2}{k^4}-1}\right) = \pm\left(\theta+\theta_0\right) \ . \tag{21}$$

Making use of the identity $\cos\left[\arctan\left(f(x)\right)\right] = \left[1+f(x)^2\right]^{-1/2}$ we may take the cosine of both sides of the above equation, yielding:

$$\sqrt{\frac{k^4}{r^2}} = \cos(\theta+\theta_0) \ , \tag{22}$$

which may be finally written as

$$r\cos(\theta+\theta_0) = k^2 \ , \tag{23}$$

which is precisely the equation of a straight line in polar co-ordinates. Thus the geodesic equations of motion, which we derived in the first part of the question, are straight lines.

Exercise 2

Consider the metric describing the two-dimensional spacetime covered by co-ordinates (t, x) and with metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 - \mathrm{d}t^2}{t^2} \ . \tag{24}$$

- Compute the Christoffel symbols.
- Compute the geodesic curves of this spacetime.

Solution 2

- As in question 1, let us first start by writing down the metric components and their partial derivatives. First consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} -t^{-2} & 0 \\ 0 & t^{-2} \end{pmatrix} \ , \tag{25}$$

$$g^{\mu\nu} = \begin{pmatrix} -t^{2} & 0 \\ 0 & t^{2} \end{pmatrix} \ , \tag{26}$$

$$g_{\mu\nu,t} = \begin{pmatrix} 2t^{-3} & 0 \\ 0 & -2t^{-3} \end{pmatrix} \ , \tag{27}$$

$$g_{\mu\nu,x} = 0 \ . \tag{28}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \ . \tag{29}$$

We may now write:

$$\Gamma^{t}_{\beta\gamma} = \frac{1}{2} g^{tt} \left(g_{t\beta,\gamma} + g_{\gamma t,\beta} - g_{\beta\gamma,t} \right) \ , \tag{30}$$

$$\Gamma^{x}_{\beta\gamma} = \frac{1}{2} g^{xx} \left(g_{x\beta,\gamma} + g_{\gamma x,\beta} - \cancel{g_{\beta\gamma,x}} \right) \ . \tag{31}$$

For equation (30) only $\beta = \gamma = t$ or x yields non-zero terms, and for equation (31) only $\beta = x$, $\gamma = t$ (or vice-versa) result in non-vanishing terms. It immediately follows that the only non-zero Christoffel symbols are all identical and are given by:

$$\Gamma^{t}_{tt} = \Gamma^{t}_{xx} = \Gamma^{x}_{tx} = -\frac{1}{t} \ . \tag{32}$$

- For the second part of the question let us first write the geodesic equations of motion for this spacetime. As in exercise 1, an overdot denotes differentiation with respect to the affine parameter. With the Christoffel symbol components we may write the geodesic equations of motion as:

$$\ddot{t} = \frac{1}{t} \left(\dot{t}^2 + \dot{x}^2 \right) \ , \tag{33}$$

$$\ddot{x} = \frac{2}{t} \dot{t} \dot{x} \ , \tag{34}$$

and from the line element we may write

$$\dot{x}^2 - \dot{t}^2 = t^2 \ . \tag{35}$$

We may write equation (34) as

$$\frac{\ddot{x}}{\dot{x}} = 2 \frac{\dot{t}}{t} \ , \tag{36}$$

which may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\ln \dot{x} \right) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\ln t^2 \right) \ . \tag{37}$$

Integrating both sides of this equation then yields

$$\dot{x} = k t^2 \ , \tag{38}$$

where k is a constant of integration. Substituting equation (38) into equation (35) yields

$$k^2 t^4 - \dot{t}^2 = t^2 \ , \tag{39}$$

which may be solved for $\dot{t}$ to yield

$$\dot{t} = \pm t \sqrt{k^2 t^2 - 1} \ . \tag{40}$$

We may now obtain a differential equation for x as a function of t by dividing equation (38) by equation (40), yielding

$$x'(t) = \pm \frac{k t}{\sqrt{k^2 t^2 - 1}} \ , \tag{41}$$

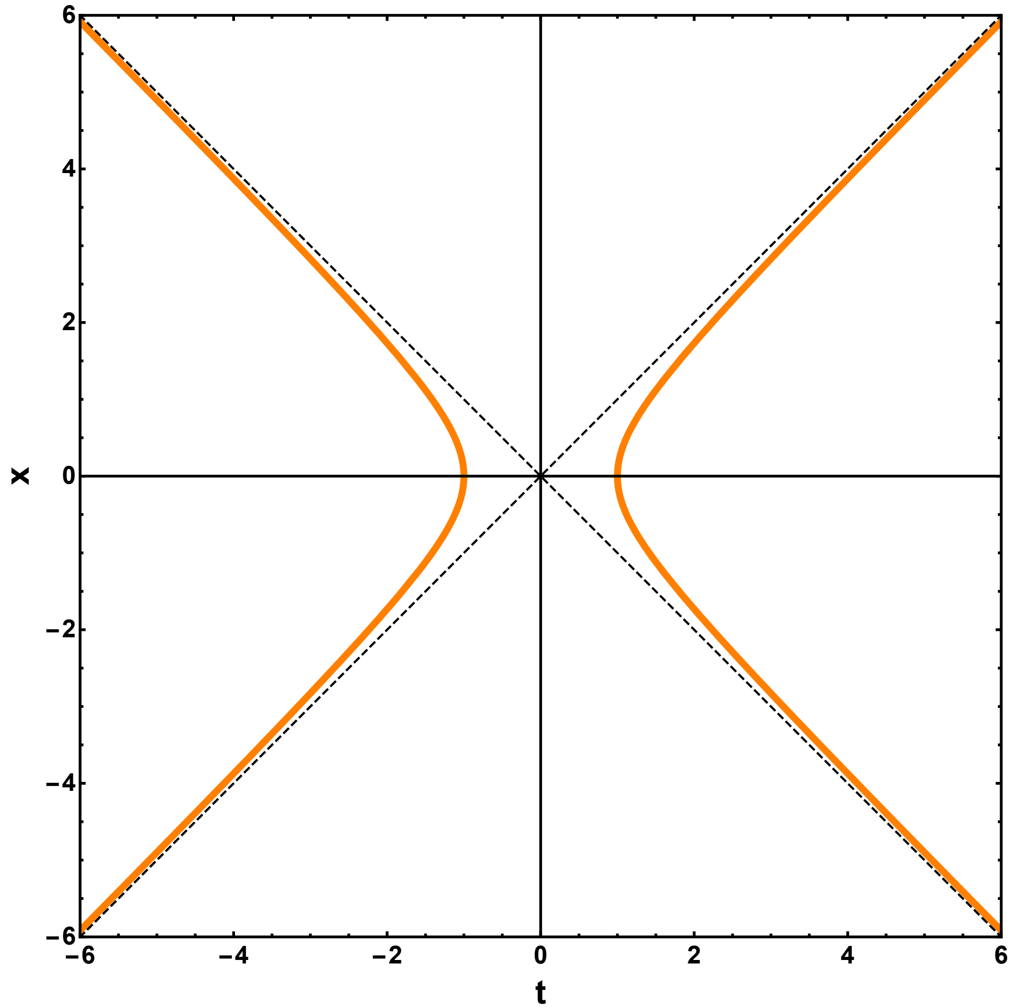

Figure 1: Hyperbolic geodesics as described by equation (43) for the case $x_0 = 0$ and $k = 1$. Note that the geodesics (orange curves) are asymptotic to the lightcone (dashed black line).

which may be integrated to give

$$\begin{aligned} x - x_0 &= \pm\frac{1}{k}\sqrt{k^2t^2 - 1} \\ &= \pm\sqrt{t^2 - k^{-2}}\ , \end{aligned} \tag{42}$$

where x_0 is a constant of integration. Finally, upon squaring both sides and rearranging we obtain

$$\frac{t^2}{(1/k)^2} - \frac{(x - x_0)^2}{(1/k)^2} = 1\ , \tag{43}$$

which is the equation of a hyperbola. Thus the geodesics curves in this spacetime are described by hyperbolas. This is illustrated in Figure 1.

Exercise 3

Given a scalar function $\phi \equiv \phi\left(x^\mu\right)$, prove the following identity in a co-ordinate basis:

$$\Box\phi := \nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \phi\right) \ . \tag{44}$$

Solution 3

First, consider the following

$$\begin{aligned} \nabla^\mu \nabla_\mu \phi &= \left(g^{\mu\nu} \phi_{;\nu}\right)_{;\mu} \\ &= \left(g^{\mu\nu} \phi_{,\nu}\right)_{;\mu} \ , \end{aligned} \tag{45}$$

since ϕ is a scalar quantity. Recall the identity we derived in Problem Sheet 7, question 3, part 5:

$$A^\mu{}_{;\mu} = \frac{1}{|g|^{1/2}} \left(|g|^{1/2} A^\mu\right)_{,\mu} \ . \tag{46}$$

Using this identity and substituting $A^\mu = g^{\mu\nu} \phi_{,\nu}$, we may now write

$$\nabla^\mu \nabla_\mu \phi = \frac{1}{|g|^{1/2}} \left(|g|^{1/2} g^{\mu\nu} \phi_{,\nu}\right)_{,\mu} \ , \tag{47}$$

which is the desired result, as required. Note that we use $|g|^{1/2}$ and $\sqrt{-g}$ interchangeably.

Exercise 4

Optional: Derive the geodesic equation from the definition of a curve of extremal length.

Solution 4

The Euler-Lagrange equations of motion are derived by extremising the length of a curve. For a given metric tensor $g_{\mu\nu}$ the Lagrangian may be written as

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \ , \tag{48}$$

where, as before, an overdot denotes differentiation with respect to the affine parameter. The Euler-Lagrange equations are by definition written as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) = \frac{\partial \mathcal{L}}{\partial x^\alpha} \ . \tag{49}$$

Let us now derive each term. First we calculate the RHS of (49):

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu \ . \tag{50}$$

For the LHS of equation (49) first consider:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} &= \frac{1}{2} g_{\mu\nu} \left(\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\alpha} \right) \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \left(\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\alpha} \right) \\ &= \frac{1}{2} g_{\mu\nu} \delta^\mu_\alpha \dot{x}^\nu + \frac{1}{2} g_{\mu\nu} \delta^\mu_\alpha \dot{x}^\mu \\ &= \frac{1}{2} g_{\alpha\nu} \dot{x}^\nu + \frac{1}{2} g_{\mu\alpha} \dot{x}^\mu \\ &= g_{\alpha\mu} \dot{x}^\mu \ , \end{aligned} \tag{51}$$

where in the last step with have made use of the fact that μ and ν are dummy indices, as well as the metric tensor being symmetric. Now we differentiate with respect to the affine parameter, yielding:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} \right) &= \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(g_{\alpha\mu} \right) \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \\ &= \dot{x}^\beta \frac{\partial}{\partial x^\beta} \left(g_{\alpha\mu} \right) \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \\ &= g_{\alpha\mu,\beta} \dot{x}^\beta \dot{x}^\mu + g_{\alpha\mu} \ddot{x}^\mu \ . \end{aligned} \tag{52}$$

Note that the dummy indices β and μ in the first term in equation (52) enable us to expand this term as follows:

$$g_{\alpha\mu,\beta} \dot{x}^\beta \dot{x}^\mu = \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} \right) \dot{x}^\beta \dot{x}^\mu \ . \tag{53}$$

Using equation (53) we may write the Euler-Lagrange equations as:

$$g_{\alpha\mu} \ddot{x}^\mu + \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} \right) \dot{x}^\beta \dot{x}^\mu = \frac{1}{2} g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu \ . \tag{54}$$

Bringing all terms to the LHS and relabelling the dummy indices μ and ν in the RHS of equation (54) as β and μ respectively, we obtain

$$g_{\alpha\mu} \ddot{x}^\mu + \frac{1}{2} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\beta\mu,\alpha} \right) \dot{x}^\beta \dot{x}^\mu = 0. \tag{55}$$

Next, multiply both sides of this expression by $g^{\delta\alpha}$, yielding

$$\ddot{x}^\delta + \frac{1}{2} g^{\delta\alpha} \left(g_{\alpha\mu,\beta} + g_{\alpha\beta,\mu} - g_{\beta\mu,\alpha} \right) \dot{x}^\beta \dot{x}^\mu = 0 \ , \tag{56}$$

where we have used the fact that $g^{\delta\alpha} g_{\alpha\mu} = \delta^\delta_\mu$. Recalling the definition of the Christoffel symbols this expression may be written more succinctly as

$$\ddot{x}^\delta + \Gamma^\delta_{\mu\beta} \dot{x}^\mu \dot{x}^\beta = 0 \ . \tag{57}$$

Let us now relabel the dummy indices as $\delta \to \alpha$, $\mu \to \beta$ and $\beta \to \gamma$, enabling us to rewrite (57) in the more familiar form

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \ , \tag{58}$$

which is precisely the geodesic equation, as required.

General Relativity: Solutions to exercises in Lecture IX

January 29, 2018

Exercise 1

Consider a torus in a two-dimensional Euclidean space described by the spherical co-ordinate system (θ, ϕ). The line element of the torus is then given by

$$\mathrm{d}s^2 = (b + a\sin\phi)^2\,\mathrm{d}\theta^2 + a^2\mathrm{d}\phi^2 \ , \tag{1}$$

where b and a denote the torus radius and the radius of its section, respectively.

Compute the Christoffel symbol components and the non-vanishing components of the (Riemann) curvature tensor. (Hint: remember that there is only one linearly independent component of the Riemann tensor in a spacetime of dimension 2).

Solution 1

First let us consider the components of the metric and their partial derivatives:

$$\begin{aligned} g_{\mu\nu} &= \begin{pmatrix} (b + a\sin\phi)^2 & 0 \\ 0 & a^2 \end{pmatrix} \ , && (2) \\ g^{\mu\nu} &= \begin{pmatrix} (b + a\sin\phi)^{-2} & 0 \\ 0 & a^{-2} \end{pmatrix} \ , && (3) \\ g_{\mu\nu,\theta} &= 0 \ , && (4) \\ g_{\mu\nu,\phi} &= \begin{pmatrix} 2a\,(b + a\sin\phi)\cos\phi & 0 \\ 0 & 0 \end{pmatrix} \ . && (5) \end{aligned}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}\right) \ . \tag{6}$$

Since α can only be θ or ϕ and the metric is diagonal, we may proceed as follows:

$$\begin{aligned}
\Gamma^{\theta}_{\beta\gamma} &= \frac{1}{2} g^{\theta\theta} \left(g_{\theta\beta,\gamma} + g_{\gamma\theta,\beta} - \cancel{g_{\beta\gamma,\theta}} \right) \\
&= \frac{1}{2} g^{\theta\theta} g_{\theta\theta,\phi} \qquad (7) \\
&= \frac{a\cos\phi}{b + a\sin\phi} \qquad (8) \\
\Gamma^{\phi}_{\beta\gamma} &= \frac{1}{2} g^{\phi\phi} \left(\cancel{g_{\phi\beta,\gamma}} + \cancel{g_{\gamma\phi,\beta}} - g_{\beta\gamma,\phi} \right) \\
&= -\frac{1}{2} g^{\phi\phi} g_{\theta\theta,\phi} \qquad (9) \\
&= -\frac{(b + a\sin\phi)\cos\phi}{a} \ . \qquad (10)
\end{aligned}$$

It immediately follows that the only non-zero Christoffel symbols are given by:

$$\begin{aligned}
\Gamma^{\theta}_{\theta\phi} &= \frac{a\cos\phi}{b + a\sin\phi} \ , \qquad (11) \\
\Gamma^{\phi}_{\theta\theta} &= -\frac{(b + a\sin\phi)\cos\phi}{a} \ . \qquad (12)
\end{aligned}$$

Next, recall the definition of the Riemann tensor:

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\nu\alpha,\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu\alpha} \ , \tag{13}$$

which may also be written more compactly as

$$R^{\mu}{}_{\nu\alpha\beta} = \left(\Gamma^{\mu}_{\nu\beta,\alpha} + \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\nu\beta} \right) - (\alpha \leftrightarrow \beta) \ , \tag{14}$$

where $(\alpha \leftrightarrow \beta)$ denotes writing down the first term in brackets with α and β exchanged. Looking at the first term in equation (13), we know that $\Gamma^{\mu}_{\nu\beta,\alpha}$ is non-zero only if $\alpha = \phi$ (partial derivative is non-zero). Next, we are free to choose (μ, ν, β) such that the Christoffel symbol is also non-zero. This yields the choices $(\mu, \nu, \beta) = (\theta, \theta, \phi)$ or (ϕ, θ, θ). Let us take $(\mu, \nu, \beta) = (\theta, \theta, \phi)$, which yields

$$\begin{aligned}
R^{\theta}{}_{\theta\phi\phi} &= \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\theta\phi} - \Gamma^{\theta}_{\theta\phi,\phi} - \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\theta\phi} \\
&= \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\theta\phi} + \cancel{\Gamma^{\theta}_{\phi\phi}}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\theta}_{\theta\phi,\phi} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\theta\phi} - \cancel{\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi}} \\
&= \Gamma^{\theta}_{\theta\phi,\phi} + \left(\Gamma^{\theta}_{\theta\phi}\right)^2 - \Gamma^{\theta}_{\theta\phi,\phi} - \left(\Gamma^{\theta}_{\theta\phi}\right)^2 \\
&= 0 \ . \qquad (15)
\end{aligned}$$

Instead, let us now try $\alpha = \theta$ and $\beta = \phi$ in equation (13). We obtain

$$R^{\mu}{}_{\nu\theta\phi} = \cancel{\Gamma^{\mu}_{\nu\phi,\theta}} + \Gamma^{\mu}_{\rho\theta}\Gamma^{\rho}_{\nu\phi} - \Gamma^{\mu}_{\nu\theta,\phi} - \Gamma^{\mu}_{\rho\phi}\Gamma^{\rho}_{\nu\theta} \ . \tag{16}$$

Next, let us ensure the partial derivative of the Christoffel symbol does not vanish by choosing $\mu = \theta$ and $\nu = \phi$, which yields:

$$\begin{aligned}
R^{\theta}{}_{\phi\theta\phi} &= \Gamma^{\theta}_{\rho\theta}\cancel{\Gamma^{\rho}_{\phi\phi}} - \Gamma^{\theta}_{\phi\theta,\phi} - \Gamma^{\theta}_{\rho\phi}\Gamma^{\rho}_{\phi\theta} \\
&= -\Gamma^{\theta}_{\phi\theta,\phi} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta} \\
&= \frac{a\,(a + b\sin\phi)}{(b + a\sin\phi)^2} - \frac{a^2\cos^2\phi}{(b + a\sin\phi)^2} \\
&= \frac{a\sin\phi}{(b + a\sin\phi)} \ . \qquad (17)
\end{aligned}$$

Consequently we may calculate the (only) non-vanishing component of the fully-covariant Riemann curvature tensor as:

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\ &= a \sin\phi \, (b + a\sin\phi) \ . \end{aligned} \tag{18}$$

Exercise 2

Consider the two-dimensional spacetime with line element

$$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2 \ . \tag{19}$$

Compute the Christoffel symbols and the non-vanishing components of the curvature tensor.

Solution 2

As in question 1, let us first start by writing down the metric components and their partial derivatives. First consider the components of the metric and their partial derivatives:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -v^2 \end{pmatrix} \ , \tag{20}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -v^2 \end{pmatrix} \ , \tag{21}$$

$$g_{\mu\nu,v} = \begin{pmatrix} 0 & 0 \\ 0 & -2v \end{pmatrix} \ , \tag{22}$$

$$g_{\mu\nu,u} = 0 \ . \tag{23}$$

Next, recall the definition of the Christoffel symbols:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right) \ . \tag{24}$$

We may calculate the Christoffel symbols as before:

$$\begin{aligned} \Gamma^{v}_{\beta\gamma} &= \frac{1}{2} g^{vv} \left(\cancel{g_{v\beta,\gamma}} + \cancel{g_{\gamma v,\beta}} - g_{\beta\gamma,v} \right) \\ &= -\frac{1}{2} g^{vv} g_{\beta\gamma,v} \\ &= -\frac{1}{2} g^{vv} g_{uu,v} \\ &= -\frac{1}{2} (1)(-2v) \\ &= v \ , \end{aligned} \tag{25}$$

$$
\begin{aligned}
\Gamma^{u}_{\beta\gamma} &= \frac{1}{2}g^{uu}\left(g_{u\beta,\gamma}+g_{\gamma u,\beta}-\cancel{g_{\beta\gamma,u}}\right) \\
&= \frac{1}{2}g^{uu}\left(g_{u\beta,\gamma}+g_{\gamma u,\beta}\right) \\
&= \frac{1}{2}g^{uu}g_{uu,v} \\
&= \frac{1}{2}\left(-\frac{1}{v^2}\right)(-2v) \\
&= \frac{1}{v}\ . \qquad (26)
\end{aligned}
$$

Thus we obtain the only non-zero Christoffel symbols as:

$$
\Gamma^{v}_{uu} = v\ , \qquad (27)
$$

$$
\Gamma^{u}_{uv} = \frac{1}{v}\ . \qquad (28)
$$

Next recall the Riemann curvature tensor as defined in equation (13). Let us first make the first term vanish by choosing $\alpha = u$, yielding

$$
R^{\mu}{}_{\nu u\beta} = \cancel{\Gamma^{\mu}_{\nu\beta,u}} + \Gamma^{\mu}_{\rho u}\Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\nu u,\beta} - \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\nu u}\ . \qquad (29)
$$

Now let us expand the sum over the dummy indices ρ:

$$
R^{\mu}{}_{\nu u\beta} = \Gamma^{\mu}_{vu}\Gamma^{v}_{\nu\beta} + \Gamma^{\mu}_{uu}\Gamma^{u}_{\nu\beta} - \Gamma^{\mu}_{\nu u,\beta} - \Gamma^{\mu}_{v\beta}\Gamma^{v}_{\nu u} - \Gamma^{\mu}_{u\beta}\Gamma^{u}_{\nu u}\ . \qquad (30)
$$

Next, let us focus on ensuring the $\Gamma^{\mu}_{\nu u,\beta}$ term is non-vanishing, which requires us to set $\beta = v$:

$$
R^{\mu}{}_{\nu uv} = \Gamma^{\mu}_{vu}\Gamma^{v}_{\nu v} + \Gamma^{\mu}_{uu}\Gamma^{u}_{\nu v} - \Gamma^{\mu}_{\nu u,v} - \cancel{\Gamma^{\mu}_{vv}}\Gamma^{v}_{\nu u} - \Gamma^{\mu}_{uv}\Gamma^{u}_{\nu u}\ . \qquad (31)
$$

From equation (31) let us first consider $\mu = u$ and $\nu = v$:

$$
\begin{aligned}
R^{u}{}_{vuv} &= \Gamma^{u}_{vu}\cancel{\Gamma^{v}_{vv}} + \cancel{\Gamma^{u}_{uu}\Gamma^{u}_{vv}} - \Gamma^{u}_{vu,v} - \Gamma^{u}_{uv}\Gamma^{u}_{vu} \\
&= -\Gamma^{u}_{vu,v} - \left(\Gamma^{u}_{uv}\right)^2 \\
&= \frac{1}{v^2} - \frac{1}{v^2} \\
&= 0\ . \qquad (32)
\end{aligned}
$$

Let us next (and finally) consider the case where $\mu = v$ and $\nu = u$:

$$
\begin{aligned}
R^{v}{}_{uuv} &= \cancel{\Gamma^{v}_{vu}\Gamma^{v}_{uv}} + \Gamma^{v}_{uu}\Gamma^{u}_{uv} - \Gamma^{v}_{uu,v} - \cancel{\Gamma^{v}_{uv}\Gamma^{u}_{uu}} \\
&= \Gamma^{v}_{uu}\Gamma^{u}_{uv} - \Gamma^{v}_{uu,v} \\
&= v\left(\frac{1}{v}\right) - 1 \\
&= 0\ . \qquad (33)
\end{aligned}
$$

We may conclude that for this particular spacetime the Riemann tensor vanishes everywhere. As such, we may say that our spacetime is flat.

Exercise 3

Consider a geodesic curve $\mathcal{C}$ and its tangent vector $\mathbf{V}$. Compute the expression for the second convective derivative of a vector field $\mathbf{A}$ along $\mathcal{C}$, i.e. an explicit expression in component form of

$$\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} \ . \tag{34}$$

Recast the resulting expressions in terms of tensors that you have already encountered and interpret the results.

Solution 3

First we must calculate the action of the convective derivative on $\mathbf{A}$:

$$\begin{aligned} \nabla_{\mathbf{V}}\mathbf{A} &= V^{\mu}\nabla_{\mu}A^{\alpha} \\ &= V^{\mu}\left(\partial_{\mu}A^{\alpha} + \Gamma^{\alpha}_{\mu\beta}A^{\beta}\right) \ . \end{aligned} \tag{35}$$

At this point it is important to remark that equation (35) is actually a rank-1 contravariant tensor which we may call T^{α} (all other indices are dummy indices). With this in mind, we may write the second convective derivative of $\mathbf{A}$ as

$$\begin{aligned} \nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} &= \nabla_{\mathbf{V}}\left(\nabla_{\mathbf{V}}\mathbf{A}\right) \\ &= V^{\nu}\nabla_{\nu}T^{\alpha} \\ &= V^{\nu}\left(\partial_{\nu}T^{\alpha} + \Gamma^{\alpha}_{\nu\rho}T^{\rho}\right) \ . \end{aligned} \tag{36}$$

From here we must explicitly expand (36), yielding:

$$\begin{aligned} \nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\partial_{\mu}A^{\alpha}\right) + V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} + \left(\partial_{\nu}V^{\mu}\right)\Gamma^{\alpha}_{\mu\beta}A^{\beta} \\ & + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta} \\ & + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\partial_{\mu}A^{\rho} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \ . \end{aligned} \tag{37}$$

Let us now write the above expression as:

$$\begin{aligned} \nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\partial_{\mu}A^{\alpha}\right) + V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} + \left(\partial_{\nu}V^{\mu}\right)\Gamma^{\alpha}_{\mu\beta}A^{\beta} \\ & + V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta} + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\partial_{\mu}A^{\rho}\Big] \\ & + V^{\nu}\Big[V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \\ = \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\partial_{\mu}A^{\alpha}\right) + V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} + \left(\partial_{\nu}V^{\mu}\right)\Gamma^{\alpha}_{\mu\beta}A^{\beta} \\ & + V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta} + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\partial_{\mu}A^{\rho}\Big] \\ & + V^{\nu}\Big[V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta} + \Gamma^{\alpha}_{\nu\rho}V^{\mu}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \ , \end{aligned} \tag{38}$$

which may be further simplified as:

$$
\begin{aligned}
\nabla_{\mathbf{V}}\nabla_{\mathbf{V}}\mathbf{A} = \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\partial_{\mu}A^{\alpha}+\Gamma^{\alpha}_{\mu\beta}A^{\beta}\right)+V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} \\
& + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta}+V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta}+V^{\mu}\Gamma^{\alpha}_{\nu\rho}\partial_{\mu}A^{\rho}\Big] \\
& + V^{\nu}\Big[V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta}+\Gamma^{\alpha}_{\nu\rho}V^{\mu}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big] \\
= \quad & V^{\nu}\Big[\left(\partial_{\nu}V^{\mu}\right)\left(\nabla_{\mu}A^{\alpha}\right)+V^{\mu}\partial_{\nu}\partial_{\mu}A^{\alpha} \\
& + V^{\mu}\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta}+2V^{\mu}\Gamma^{\alpha}_{\mu\beta}\partial_{\nu}A^{\beta}\Big] \quad \text{(letting } \rho\to\beta \text{ and } \mu\leftrightarrow\nu\text{)} \\
& + V^{\nu}V^{\mu}\Big[\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta}+\Gamma^{\alpha}_{\nu\rho}\Gamma^{\rho}_{\mu\beta}A^{\beta}\Big]\ . \qquad (39)
\end{aligned}
$$

At this point we remark that all terms in the first square brackets of equation (39) are unchanged under interchange of μ and ν indices, whereas the two terms in the second pair of square brackets are not. As such, if we calculate $2\nabla_{[\nu}\nabla_{\mu]}A^{\alpha}$ we will find that the first set of terms in the square brackets will vanish. Doing this for the second convective derivative we derived we obtain:

$$
\begin{aligned}
2\nabla_{[\mathbf{V}}\nabla_{\mathbf{V}]}\mathbf{A} &= V^{\nu}V^{\mu}\left(\partial_{\nu}\Gamma^{\alpha}_{\mu\beta}A^{\beta}+\Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta}A^{\beta}-\partial_{\mu}\Gamma^{\alpha}_{\nu\beta}A^{\beta}-\Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\nu\beta}A^{\beta}\right) \\
&= V^{\nu}V^{\mu}R^{\alpha}{}_{\beta\nu\mu}A^{\beta}\ , \qquad (40)
\end{aligned}
$$

thus we obtain an expression which depends on the Riemann curvature tensor. The expression $\nabla_{[\mathbf{V}}\nabla_{\mathbf{V}]}$ (or $\nabla_{[\nu}\nabla_{\mu]}$ in component form) thus measure differences in a vector which is transported in different directions around (say) a closed loop but which reach the same point. This equation is known as the geodesic deviation equation.

General Relativity: Solutions to exercises in Lecture X

January 29, 2018

Exercise 1

Show that the second covariant derivatives of a scalar field commute, i.e. that

$$\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi \ . \tag{1}$$

Obtain the expressions for the following third derivatives: $\nabla_\alpha \nabla_{(\beta} \nabla_{\gamma)} \phi$ and $\nabla_{[\alpha} \nabla_{\beta]} \nabla_\gamma \phi$. [Hint: remember that the covariant derivative of a scalar field is a vector.]

Solution 1

- For the first part of the question we consider the action of the covariant derivatives in order, remembering that the covariant derivative of a scalar is simply the partial derivative acting on that scalar. This yields:

$$\begin{aligned} \nabla_\alpha \nabla_\beta \phi &= \left(\phi_{;\beta}\right)_{;\alpha} \\ &= \left(\phi_{,\beta}\right)_{;\alpha} \\ &= \phi_{,\alpha\beta} - \phi_{,\delta} \Gamma^\delta{}_{\alpha\beta} \ . \end{aligned} \tag{2}$$

  Since the Christoffel symbols are symmetric in their lower indices (torsion-free) and partial derivatives commute, we may conclude that $\nabla_\alpha \nabla_\beta \phi = \nabla_\beta \nabla_\alpha \phi$, as required.

- For the second part of the question, let us first define the covariant vector $W_\gamma \equiv \nabla_\gamma \phi$. Now consider

$$\nabla_\alpha \nabla_\beta \nabla_\gamma \phi = \nabla_\alpha \nabla_\beta W_\gamma \ , \tag{3}$$

  and similarly

$$\nabla_\alpha \nabla_\gamma \nabla_\beta \phi = \nabla_\alpha \nabla_\gamma W_\beta \ . \tag{4}$$

  Using the result of the first part of the question we may write

$$\nabla_\beta W_\gamma = \nabla_\gamma W_\beta \ . \tag{5}$$

  Employing the above we may now write

$$\begin{aligned} \nabla_\alpha \nabla_{(\beta} \nabla_{\gamma)} \phi &= \frac{1}{2} \nabla_\alpha \left(\nabla_\beta W_\gamma + \nabla_\gamma W_\beta \right) \\ &= \nabla_\alpha \nabla_\beta W_\gamma \\ &= \nabla_\alpha \nabla_\beta \nabla_\gamma \phi \ . \end{aligned} \tag{6}$$

- For the final part of the question let us directly expand the expression in question:

$$\begin{aligned}
\nabla_{[\alpha}\nabla_{\beta]}\nabla_\gamma\phi &= \nabla_{[\alpha}\nabla_{\beta]}W_\gamma \\
&= \frac{1}{2}\left(\nabla_\alpha\nabla_\beta W_\gamma - \nabla_\beta\nabla_\alpha W_\gamma\right) \\
&= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}W_\delta \\
&= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}\phi_{;\gamma} \\
&= \frac{1}{2}R^\delta{}_{\alpha\beta\gamma}\phi_{,\gamma} \ . \qquad (7)
\end{aligned}$$

Exercise 2

Prove that for any second-rank tensor, the covariant derivative commutes, i.e. that

$$\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \nabla_\beta\nabla_\alpha V^{\alpha\beta} \ . \qquad (8)$$

Solution 2

First recall the fact that $\nabla_\beta V^{\alpha\beta}$ is a rank-1 contravariant tensor, thus we may define $W^\mu \equiv \nabla_\beta V^{\mu\beta}$. Next, let us write explicitly the expression for W^μ as follows:

$$W^\mu = \partial_\beta V^{\mu\beta} + \Gamma^\mu{}_{\beta\delta}V^{\delta\beta} + \Gamma^\beta{}_{\beta\delta}V^{\mu\delta} \ . \qquad (9)$$

We may now write the covariant derivatives acting on $V^{\alpha\beta}$ as:

$$\begin{aligned}
\nabla_\alpha\nabla_\beta V^{\alpha\beta} &= \nabla_\alpha W^\alpha \\
&= \partial_\alpha W^\alpha + \Gamma^\alpha{}_{\alpha\gamma}W^\gamma \ . \qquad (10)
\end{aligned}$$

Thus we may write the LHS and RHS of equation (8) as:

$$\begin{aligned}
\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \partial_\alpha\partial_\beta V^{\alpha\beta} &+ \partial_\alpha\left(\Gamma^\alpha{}_{\beta\delta}V^{\delta\beta}\right) + \partial_\alpha\left(\Gamma^\beta{}_{\beta\delta}V^{\alpha\delta}\right) + \Gamma^\alpha{}_{\alpha\gamma}\partial_\beta V^{\gamma\beta} + \Gamma^\alpha{}_{\alpha\gamma}\Gamma^\gamma{}_{\beta\delta}V^{\delta\beta} \\
&+ \Gamma^\alpha{}_{\alpha\gamma}\Gamma^\beta{}_{\beta\delta}V^{\gamma\delta} \ , \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_\beta\nabla_\alpha V^{\alpha\beta} = \partial_\beta\partial_\alpha V^{\alpha\beta} &+ \partial_\beta\left(\Gamma^\alpha{}_{\alpha\delta}V^{\delta\beta}\right) + \partial_\beta\left(\Gamma^\beta{}_{\alpha\delta}V^{\alpha\delta}\right) + \Gamma^\beta{}_{\beta\gamma}\partial_\alpha V^{\alpha\gamma} + \Gamma^\beta{}_{\beta\gamma}\Gamma^\alpha{}_{\alpha\delta}V^{\delta\gamma} \\
&+ \Gamma^\beta{}_{\beta\gamma}\Gamma^\gamma{}_{\alpha\delta}V^{\alpha\delta} \ . \qquad (12)
\end{aligned}$$

Consider each term in eqns. (11) & (12), and let us refer to these equations as L and R respectively, along with the indices 1–6 indicating terms 1–6 respectively in each expression. Under $\alpha \leftrightarrow \beta$:

$$\begin{aligned}
L_1 &= R_1 \ , \\
L_2 &= R_3 \ , \\
L_3 &= R_2 \ , \\
L_4 &= R_4 \ , \\
L_5 &= R_6 \ , \\
L_6 &= R_5 \ .
\end{aligned}$$

We may thus conclude that $\nabla_\alpha\nabla_\beta V^{\alpha\beta} = \nabla_\beta\nabla_\alpha V^{\alpha\beta}$, as required.

Exercise 3

Optional: Find the matrix of the Lorentz transformations corresponding to a boost v^x in the x-direction followed by a boost v^y in the y-direction. What happens if the order of the boosts is reversed?

Solution 3

Let us write the Lorentz boost matrices in the x- and y-directions respectively as:

$$\Lambda_x = \begin{pmatrix} \gamma_x & \gamma_x v_x & 0 & 0 \\ \gamma_x v_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

and

$$\Lambda_y = \begin{pmatrix} \gamma_y & 0 & \gamma_y v_y & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_y v_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

For the first combination of boosts we obtain:

$$\Lambda_y \Lambda_x = \begin{pmatrix} \gamma_x \gamma_y & \gamma_x \gamma_y v_x & \gamma_y v_y & 0 \\ \gamma_x v_x & \gamma_x & 0 & 0 \\ \gamma_x \gamma_y v_y & \gamma_x \gamma_y v_x v_y & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Similarly, for the reverse transformation we find:

$$\Lambda_x \Lambda_y = \begin{pmatrix} \gamma_x \gamma_y & \gamma_x v_x & \gamma_x \gamma_y v_y & 0 \\ \gamma_x \gamma_y v_x & \gamma_x & \gamma_x \gamma_y v_x v_y & 0 \\ \gamma_y v_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Clearly $\Lambda_y \Lambda_x \neq \Lambda_x \Lambda_y$ and so the transformations do not commute.

General Relativity: Solutions to exercises in Lecture XI

January 29, 2018

All of the following exercises are to be considered in a special-relativistic context and assuming Cartesian co-ordinates where necessary.

Exercise 1

Within Special Relativity, consider a four-vector $\mathbf{V}$ with components:

$$\mathbf{V} = \sqrt{3}\,\mathbf{e}_t + \sqrt{2}\,\mathbf{e}_x \ . \tag{1}$$

Determine if $\mathbf{V}$ is timelike, null or spacelike. Compute the angles between $\mathbf{V}$ and the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_x$.

Solution 1

First let us consider the inner-producut of $\mathbf{V}$ with itself:

$$\begin{aligned}\mathbf{V}\cdot\mathbf{V} &= \left(\sqrt{3}\right)^2 \mathbf{e}_t\cdot\mathbf{e}_t + \left(\sqrt{2}\right)^2 \mathbf{e}_x\cdot\mathbf{e}_x + 2\sqrt{2}\sqrt{3}\,\mathbf{e}_t\cdot\mathbf{e}_x \\ &= -3+2+0 \\ &= -1 \ < \ 0 \ ,\end{aligned} \tag{2}$$

therefore $\mathbf{V}$ is timelike. For the angles, first let us consider the t-component. From the inner-product we may calculate the angle between $\mathbf{V}$ and $\mathbf{e}_t$ as:

$$\begin{aligned}\cos\theta_t &= \frac{\mathbf{V}\cdot\mathbf{e}_t}{|\mathbf{V}\cdot\mathbf{V}|^{1/2}|\mathbf{e}_t\cdot\mathbf{e}_t|^{1/2}} \\ &= -\sqrt{3} \ < -1 \ ,\end{aligned} \tag{3}$$

which is not satisfied for any real θ_t. Similarly, for θ_x we find:

$$\cos\theta_x = \sqrt{2} \ > \ 1 \ , \tag{4}$$

which is also not satisfied for any real θ_x.

Exercise 2

A particle with rest mass m and four-momentum $\mathbf{p} = m\mathbf{v}$ is analysed by an observer with four-velocity $\mathbf{u}$. Compute the following:

- The total energy of the particle
- The kinetic energy of the particle
- The magnitude of the spatial momentum $p := \sqrt{p_i p^i}$
- The magnitude of the three-velocity $v := \sqrt{v_i v^i}$

Solution 2

Let us work in the rest frame of the observer. In this frame:

$$\begin{aligned} u^\alpha &= (1, \underline{0}) \ , && (5) \\ u_\alpha &= (-1, \underline{0}) \ , && (6) \\ p^\alpha &= (E, \underline{p}) \ , && (7) \\ p_\alpha &= (-E, \underline{p}) \ , && (8) \end{aligned}$$

where $\underline{p}$ is the three-momentum.

- The total energy may be obtained directly as

$$\begin{aligned} E &= -p_0 u^0 \\ &= -p_\alpha u^\alpha \ . && (9) \end{aligned}$$

- Starting form the expression for the total energy of a particle as $E^2 = p^2 c^2 + m^2 c^4$ one obtains the rest mass of the particle as:

$$\begin{aligned} m^2 &= E^2 - |\underline{p}|^2 \\ &= -p_0 p^0 - p_i p^i \\ &= -p_\alpha p^\alpha \ , && (10) \end{aligned}$$

from which the kinetic energy of the particle may be directly derived as:

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} m |\underline{v}|^2 \\ &= \frac{1}{2} \sqrt{-p_\alpha p^\alpha}\, v^2 \\ &= \frac{1}{2} \sqrt{-p_\alpha p^\alpha} \sqrt{-v_i v^i} \ . && (11) \end{aligned}$$

- Starting again from the expression for the total energy of the particle, the magnitude of the three-momentum may be calculated as:

$$\begin{aligned} p &= \left(E^2 - m^2\right)^{1/2} \\ &= \left[(p_\alpha u^\alpha)^2 + p_\beta p^\beta\right]^{1/2} \ . && (12) \end{aligned}$$

- Finally, the magnitude of the three-velocity may be calculated directly as:

$$\begin{aligned} v &= \frac{p}{E} \\ &= \frac{1}{E}\left(E^2 - m^2\right)^{1/2} \\ &= \left(1 - \frac{m^2}{E^2}\right)^{1/2} \\ &= \left[1 + \frac{p_\alpha p^\alpha}{(p_\beta u^\beta)^2}\right]^{1/2} \quad . \end{aligned} \tag{13}$$

Exercise 3

Define the four-acceleration of a particle with four-velocity $\mathbf{u}$ as

$$a^\mu := \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} \ , \tag{14}$$

where τ is the proper time. Show that $\mathbf{a} \cdot \mathbf{u} = 0$, i.e. that the acceleration is orthogonal to the four-velocity. What does this mean in a frame co-moving with the particle?

Solution 3

Let us start with the following identity for a particle in General Relativity:

$$u_\mu u^\mu = -1 \ , \tag{15}$$

from which it immediately follows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) = 0 \ . \tag{16}$$

Expanding the above expression yields:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(u_\mu u^\mu\right) &= 2\,\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}\,u^\mu \\ &= 2\,a_\mu u^\mu \\ &= 0 \ . \end{aligned} \tag{17}$$

We may thus conclude that $\mathbf{a} \cdot \mathbf{u} = 0$, as required.

In a frame co-moving with the particle, $u^\mu = (1, \underline{0})$. In this frame $a_\mu u^\mu = 0$ implies that $a_0 = 0$ by necessity, but that the spatial components of the four-acceleration, a_i, are arbitrary.

General Relativity: Solutions to exercises in Lecture XII

January 29, 2018

Exercise 1

Consider the stress-energy-momentum tensor of a perfect fluid

$$T^{\mu\nu} = (e+p)u^\mu u^\nu + p\, g^{\mu\nu} \ , \tag{1}$$

and its conservation equation

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{2}$$

Show that the equations (2) lead to the Euler equations, i.e. to the equations of conservation of momentum

$$(e+p)\nabla_{\mathbf{u}}\mathbf{u} = -\left[\nabla p + (\nabla_{\mathbf{u}}\, p)\, \mathbf{u}\right] \ . \tag{3}$$

[Hint: use the projector $\mathbf{h} = \mathbf{g} + \mathbf{u}\,\mathbf{u}$]. Do equations (3) bear resemblance with the Newtonian Euler equations?

Solution 1

First, let us write out the covariant derivative of the stress-energy-momentum tensor:

$$\nabla_\mu T^{\mu\nu} = (e+p)_{;\mu} u^\mu u^\nu + (e+p)\left(u^\mu{}_{;\mu} u^\nu + u^\mu u^\nu{}_{;\mu}\right) + p_{;\mu} g^{\mu\nu} \ . \tag{4}$$

Let us now us the projection tensor $h_{\alpha\nu} = g_{\alpha\nu} + u_\alpha u_\nu$ to project orthogonally to $\mathbf{u}$, yielding

$$\begin{aligned} h_{\alpha\nu}\nabla_\mu T^{\mu\nu} &= (e+p)_{;\mu}[\cancel{u^\mu u_\alpha + u_\alpha u^\mu (u_\nu u^\nu)}] + (e+p)\big[\cancel{\big(u^\mu{}_{;\mu} u_\alpha + u_\alpha u^\mu{}_{;\mu}(u_\nu u^\nu)\big)} \\ &\quad + \left(u^\mu g_{\alpha\nu} u^\nu{}_{;\mu} + u_\alpha u^\mu u_\nu u^\nu{}_{;\mu}\right)\big] + p_{,\mu}\delta^\mu_\alpha + p_{,\mu} u^\mu u_\alpha \\ &= (e+p) u^\mu u_{\alpha;\mu} + p_{,\alpha} + p_{,\mu} u^\mu u_\alpha \ , \end{aligned} \tag{5}$$

where we have used the fact that $u_\nu u^\nu = -1$ and $u_\nu u^\nu{}_{;\mu} = 0$. Since $h_{\alpha\nu}\nabla_\mu T^{\mu\nu} = 0$ we may now write

$$(e+p) u^\mu u_{\alpha;\mu} = -\left(p_{,\alpha} + p_{,\mu} u^\mu u_\alpha\right) \ , \tag{6}$$

which is equivalent to

$$(e+p)\nabla_{\mathbf{u}}\mathbf{u} = -\left[\nabla p + (\nabla_{\mathbf{u}}\, p)\, \mathbf{u}\right] \ , \tag{7}$$

as required.

In the Newtonian limit we may adopt the following approximations:

- $p \ll e$,
- $e \approx \rho_0$,
- $v^2 \ll 1$,
- $g_{00} = -(1+2\phi)$, $|\phi| \ll 1$, where ϕ is the Newtonian potential.

We may immediately let $(e+p) \to \rho_0$, and through expanding the covariant derivative we obtain

$$\rho_0 \left[u^\mu u_{\alpha,\mu} - \Gamma^\beta{}_{\mu\alpha} u_\beta u^\mu\right] = -p_{,\alpha} - u_\alpha u^\mu p_{,\mu} \ . \tag{8}$$

We now take: (i) $u_\beta u^\mu \sim O(v^2)$ for $\beta \neq \mu$ and $u_\beta u^\mu = -1$ for $\beta = \mu$ and (ii) $u_\alpha u^\mu p_{,\mu} \sim O(v^2) \to 0$. With these in mind we obtain

$$\rho_0 \left[u^\mu u_{\alpha,\mu} + \Gamma^\beta{}_{\beta\alpha}\right] = -p_{,\alpha} \ . \tag{9}$$

Recall from Problem Sheet 7, Exercise 3, part 5 we derived the following expression:

$$\Gamma^\beta{}_{\beta\alpha} = \frac{1}{2} g^{\beta\delta} g_{\delta\beta,\alpha} \ . \tag{10}$$

Since spacetime is now flat and $g_{00} = -(1+2\phi)$, so $g^{00} = -(1/g_{00}) \approx -1$, since $|\phi| \ll 1$. This implies

$$\begin{aligned}
\Gamma^\beta{}_{\beta\alpha} &= \Gamma^0{}_{0\alpha} \\
&\approx \frac{1}{2}(-1)\left[-(1+2\phi)\right]_{,\alpha} \\
&= \phi_{,\alpha} \ ,
\end{aligned} \tag{11}$$

hence we obtain

$$\rho_0 \left(u^\mu u_{\alpha,\mu} + \phi_{,\alpha}\right) = -p_{,\alpha} \ , \tag{12}$$

which may be rewritten as

$$u^\mu u_{\alpha,\mu} = -\frac{1}{\rho_0} p_{,\alpha} - \phi_{,\alpha} \ , \tag{13}$$

which is precisely the (Newtonian) incompressible Euler momentum equation with a constant and uniform density. This may be written more succinctly as follows. First define the specific thermodynamic work, w, where $w \equiv p/\rho_0$ and the gravitational acceleration $\mathbf{g} \equiv -\nabla\phi$. The material derivative is defined in general relativity as $\frac{\mathrm{D}}{\mathrm{D}\tau} = u^\mu \nabla_\mu$ and in the Newtonian regime as $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$. We may now rewrite equation (13) as

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla w + \mathbf{g} \ . \tag{14}$$

Exercise 2

The stress-energy-momentum tensor of a scalar field Φ is defined as

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\partial_\mu \Phi \, \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \, \partial_\alpha \Phi \, \partial^\alpha \Phi\right) \ . \tag{15}$$

Derive the expression for the conservation of energy and momentum (2) in this case. Interpret the results.

Solution 2

We first write out the covariant derivative $\nabla^\mu T_{\mu\nu} = 0$ as follows

$$\nabla^\mu \left(4\pi T_{\mu\nu}\right) = \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \Phi_{,\mu}\Phi_{,\nu}{}^{;\mu} - \frac{1}{2}g_{\mu\nu}\left(\Phi_{,\alpha}\Phi^{,\alpha}\right)^{;\mu} \ . \tag{16}$$

Defining the differential portion of the third term as $\Delta = \left(\Phi_{,\alpha}\Phi^{,\alpha}\right)^{;\mu}$ we may write

$$\begin{aligned}
\Delta &= \Phi_{,\alpha}{}^{;\mu}\Phi^{,\alpha} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= g^{\alpha\beta}g_{\alpha\gamma}\Phi^{,\gamma;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= \delta^\beta_\gamma \Phi^{,\gamma;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= \Phi^{,\beta;\mu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi^{,\alpha;\mu} \\
&= 2\Phi_{,\alpha}\Phi^{,\alpha;\mu} \ .
\end{aligned} \tag{17}$$

Equation (16) becomes

$$\begin{aligned}
\nabla^\mu \left(4\pi T_{\mu\nu}\right) &= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \Phi_{,\mu}\Phi_{,\nu}{}^{;\mu} - \Phi_{,\alpha}\Phi^{,\alpha}{}_{;\nu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu}{}^{;\mu} - \Phi^{,\mu}{}_{;\nu}\right)\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(g^{\alpha\mu}\Phi_{,\nu;\alpha} - g^{\beta\mu}\Phi_{,\beta;\nu}\right)\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu;\alpha} - \Phi_{,\alpha;\nu}\right)g^{\alpha\mu}\Phi_{,\mu} \\
&= \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} + \left(\Phi_{,\nu;\alpha} - \Phi_{,\alpha;\nu}\right)\Phi^{,\alpha} \ .
\end{aligned} \tag{18}$$

Since partial derivatives commute, and the covariant derivative of a scalar is simply the partial derivative, the second term in brackets vanishes and we may write

$$\nabla^\mu \left(4\pi T_{\mu\nu}\right) = \Phi_{,\mu}{}^{;\mu}\Phi_{,\nu} \ . \tag{19}$$

Since we assume $\Phi_{,\nu} \neq 0$ and $\Phi_{,\mu}{}^{;\mu} \equiv \Phi_{;\mu}{}^{;\mu}$ we may write the conservation of energy and momentum as

$$\Phi_{;\mu}{}^{;\mu} = 0 \ , \tag{20}$$

which is equivalent to

$$\Box\Phi = 0 \ . \tag{21}$$

Thus $\mathbf{\Phi}$ satisfies the wave equation for a scalar field in vacuum.

Exercise 3

Show that the Einstein equations in vacuum reduce to

$$R_{\mu\nu} = 0 \ . \tag{22}$$

Solution 3

Let us start from the definition of the Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} \ . \tag{23}$$

In the presence of matter the Einstein field equations, $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ may be written as

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu} \ . \tag{24}$$

Multiplying both sides of this equation by $g^{\mu\nu}$ yields

$$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu} g_{\mu\nu} = 8\pi g^{\mu\nu} T_{\mu\nu} \ , \tag{25}$$

which simplifies to

$$R = -4\pi T \ , \tag{26}$$

where we have used the fact that $g^{\mu\nu} g_{\mu\nu} = 4$ and defined $T \equiv g^{\mu\nu} T_{\mu\nu}$. Substituting $R = -4\pi T$ back into equation (24) yields, upon simplification

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \ . \tag{27}$$

In vacuum $T_{\mu\nu} = 0$, which implies $T = 0$, and thus we obtain

$$R_{\mu\nu} = 0 \ , \tag{28}$$

as required.

General Relativity: Solutions to exercises in Lecture XIII

January 29, 2018

Exercise 1

Consider the spherically symmetric static line element

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + C(r)\,\mathrm{d}\Omega^2\ , \tag{1}$$

and compute the expressions for the non-zero Christoffel symbols. Use this result to compute the 00 covariant component of the Einstein equations in vacuum, i.e. $R_{\mu\nu} = 0$.

Solution 1

- Whilst one may calculate the Christoffel symbol components directly, we will derive them from the Lagrangian for the metric. First let us write the Lagrangian as

$$\mathcal{L} = \frac{1}{2}\left(-A\,\dot{t}^2 + B\,\dot{r}^2 + C\,\dot{\theta}^2 + C\,\sin^2\theta\,\dot{\phi}^2\right)\ , \tag{2}$$

where the dependence of A, B and C on r has been omitted for brevity and an overdot denotes differentiation with respect to the affine parameter, λ. We now systematically derive the Euler-Lagrange equations of motion for each of the four components of our metric. For the t component:

$$\frac{\partial\mathcal{L}}{\partial t} = 0\ , \tag{3}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{t}} = -A\,\dot{t}\ , \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\dot{t}}\right) = -A'\,\dot{r}\,\dot{t} - A\,\ddot{t}\ , \tag{5}$$

where primed quantities denote differentiation with respect to r. Thus from the Euler-Lagrange equations we obtain the geodesic equation of motion for t as

$$\ddot{t} = -\left(\frac{A'}{A}\right)\dot{r}\,\dot{t}\ . \tag{6}$$

This may be immediately compared to the geodesic equation of motion for t, yielding the non-zero Christoffel symbol components as

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{1}{2}\left(\frac{A'}{A}\right) \; . \tag{7}$$

Next we consider the r component of the Euler-Lagrange equations, yielding

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{1}{2}\left(-A'\,\dot{t}^2 + B'\,\dot{r}^2 + C'\,\dot{\theta}^2 + C'\,\sin^2\theta\,\dot{\phi}^2\right) \; , &(8)\\
\frac{\partial \mathcal{L}}{\partial \dot{r}} &= B\,\dot{r}\,\dot{t} \; , &(9)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) &= B'\,\dot{r}^2 + B\,\ddot{r} \; . &(10)
\end{aligned}$$

We may now write the geodesic equation of motion for r as

$$\ddot{r} = -\frac{1}{2}\left(\frac{A'}{B}\right)\dot{t}^2 - \frac{1}{2}\left(\frac{B'}{B}\right)\dot{r}^2 + \frac{1}{2}\left(\frac{C'}{B}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{C'}{B}\right)\sin^2\theta\,\dot{\phi}^2 \; , \tag{11}$$

from which we directly obtain the Christoffel symbols as

$$\begin{aligned}
\Gamma^r{}_{tt} &= \frac{1}{2}\left(\frac{A'}{B}\right) \; , &(12)\\
\Gamma^r{}_{rr} &= \frac{1}{2}\left(\frac{B'}{B}\right) \; , &(13)\\
\Gamma^r{}_{\theta\theta} &= -\frac{1}{2}\left(\frac{C'}{B}\right) \; , &(14)\\
\Gamma^r{}_{\phi\phi} &= -\frac{1}{2}\left(\frac{C'}{B}\right)\sin^2\theta \; . &(15)
\end{aligned}$$

Now considering the θ component of the Euler-Lagrange equations we obtain

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= C\,\sin\theta\,\cos\theta\,\dot{\phi}^2 \; , &(16)\\
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= C\,\dot{\theta}\,\dot{t} \; , &(17)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= C'\,\dot{r}\,\dot{\theta} + C\,\ddot{\theta} \; . &(18)
\end{aligned}$$

We may now write the geodesic equation of motion for θ as

$$\ddot{\theta} = -\left(\frac{C'}{C}\right)\dot{r}\,\dot{\theta} + \sin\theta\,\cos\theta\,\dot{\phi}^2 \; , \tag{19}$$

from which we directly obtain the Christoffel symbols as

$$\begin{aligned}
\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{2}\left(\frac{C'}{C}\right) \; , &(20)\\
\Gamma^\theta{}_{\phi\phi} &= -\sin\theta\cos\theta \; . &(21)
\end{aligned}$$

Finally, we consider the ϕ component of the Euler-Lagrange equations, obtaining

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \ , \tag{22}$$
$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = C \sin^2\theta\, \dot t\, \dot\phi \ , \tag{23}$$
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = C' \sin^2\theta\, \dot r\, \dot\phi + C \sin 2\theta\, \dot\theta\, \dot\phi + C \sin^2\theta\, \ddot\phi \ . \tag{24}$$

We may now write the geodesic equation of motion for ϕ as

$$\ddot\phi = -\left(\frac{C'}{C}\right)\dot r\, \dot\theta - 2\cot\theta\, \dot\theta\, \dot\phi \ , \tag{25}$$

from which we directly obtain the remaining non-zero Christoffel symbols as

$$\Gamma^{\phi}{}_{r\phi} = \Gamma^{\phi}{}_{\phi r} = \frac{1}{2}\left(\frac{C'}{C}\right) \ , \tag{26}$$
$$\Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \cot\theta \ . \tag{27}$$

- For the second part of the question, recall the definition of the Riemann curvature tensor

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\alpha}{}_{\mu\gamma} - \Gamma^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta} \ . \tag{28}$$

The Ricci tensor is then defined as

$$\begin{aligned} R_{\beta\delta} &= R^{\alpha}{}_{\beta\alpha\delta} \\ &= \Gamma^{\alpha}{}_{\beta\delta,\alpha} - \Gamma^{\alpha}{}_{\beta\alpha,\delta} + \Gamma^{\mu}{}_{\beta\delta}\Gamma^{\alpha}{}_{\mu\alpha} - \Gamma^{\mu}{}_{\beta\alpha}\Gamma^{\alpha}{}_{\mu\delta} \ . \end{aligned} \tag{29}$$

The covariant 00 component may now be written as

$$\begin{aligned} R_{00} &= \Gamma^{\alpha}{}_{00,\alpha} - \cancelto{0}{\Gamma^{\alpha}{}_{0\alpha,0}} + \Gamma^{\mu}{}_{00}\Gamma^{\alpha}{}_{\mu\alpha} - \Gamma^{\mu}{}_{0\alpha}\Gamma^{\alpha}{}_{\mu 0} \\ &= \Gamma^{\alpha}{}_{00,\alpha} + \Gamma^{\mu}{}_{00}\Gamma^{\alpha}{}_{\mu\alpha} - \Gamma^{\mu}{}_{0\alpha}\Gamma^{\alpha}{}_{\mu 0} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\Gamma^{\alpha}{}_{r\alpha} - \Gamma^{\mu}{}_{0\alpha}\Gamma^{\alpha}{}_{\mu 0} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\Gamma^{\alpha}{}_{r\alpha} - \Gamma^{\mu}{}_{00}\Gamma^{0}{}_{\mu 0} - \Gamma^{\mu}{}_{0r}\Gamma^{r}{}_{\mu 0} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\Gamma^{\alpha}{}_{r\alpha} - \Gamma^{r}{}_{00}\Gamma^{0}{}_{r0} - \Gamma^{0}{}_{0r}\Gamma^{r}{}_{00} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\Gamma^{\alpha}{}_{r\alpha} - 2\Gamma^{r}{}_{00}\Gamma^{0}{}_{r0} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\left(\Gamma^{0}{}_{r0} + \Gamma^{r}{}_{rr} + \Gamma^{\theta}{}_{r\theta} + \Gamma^{\phi}{}_{r\phi}\right) - 2\Gamma^{r}{}_{00}\Gamma^{0}{}_{r0} \\ &= \Gamma^{r}{}_{00,r} + \Gamma^{r}{}_{00}\left(\Gamma^{r}{}_{rr} + \Gamma^{\theta}{}_{r\theta} + \Gamma^{\phi}{}_{r\phi} - \Gamma^{0}{}_{r0}\right) \ . \end{aligned} \tag{30}$$

Substituting the values for the Christoffel symbol components into equation (30) we obtain, upon simplification

$$R_{00} = \frac{1}{2}\frac{A''}{B} + \frac{1}{4}\frac{A'}{B}\left[2\frac{C'}{C} - \frac{A'}{A} - \frac{B'}{B}\right] \ . \tag{31}$$

For completeness, the remaining non-zero covariant components of the Ricci tensor are

$$R_{11} = -\frac{1}{2}\frac{A''}{A} - \frac{C''}{C} + \frac{1}{4}\frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{2}\frac{C'}{C}\left(\frac{C'}{C} + \frac{B'}{B}\right) , \tag{32}$$

$$R_{22} = 1 - \frac{1}{2}\frac{C''}{B} + \frac{1}{4}\frac{C'}{B}\left(\frac{B'}{B} - \frac{A'}{A}\right) , \tag{33}$$

$$R_{33} = R_{22}\sin^2\theta . \tag{34}$$

Exercise 2

Using the Lagrangian

$$2\mathcal{L} = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta , \tag{35}$$

where an overdot corresponds to differentiation with respect to the proper time, show that the geodesic equations

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0 , \tag{36}$$

are equivalent to the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha}\right) = 0 . \tag{37}$$

Solution 2

Let us first calculate the first term in equation (37):

$$\frac{\partial\mathcal{L}}{\partial x^\gamma} = \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta . \tag{38}$$

Now we consider the bracketed second term:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{x}^\gamma} &= \frac{1}{2}g_{\alpha\beta}\left(\delta^\alpha_\gamma\dot{x}^\beta + \dot{x}^\alpha\delta^\beta_\gamma\right) \\ &= \frac{1}{2}\left(g_{\gamma\beta}\dot{x}^\beta + \dot{x}^\alpha g_{\alpha\gamma}\right) \\ &= g_{\alpha\gamma}\dot{x}^\alpha .\end{aligned} \tag{39}$$

Finally, we differentiate equation (39) with respect to proper time, yielding:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\gamma}\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\gamma}\right)\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha \\ &= \dot{x}^\delta g_{\alpha\gamma,\delta}\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha \\ &= g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha + g_{\alpha\gamma}\ddot{x}^\alpha .\end{aligned} \tag{40}$$

Now we may write down the Euler-Lagrange equations, and solving for $\ddot{x}^\alpha$ we obtain

$$\begin{aligned}g_{\alpha\gamma}\ddot{x}^\alpha &= \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta - g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha \\ &= \frac{1}{2}g_{\alpha\beta,\gamma}\dot{x}^\alpha\dot{x}^\beta - \frac{1}{2}\left(g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\alpha + g_{\delta\gamma,\alpha}\dot{x}^\delta\dot{x}^\alpha\right) ,\end{aligned} \tag{41}$$

where we have made use of the symmetry under interchange of $\delta \leftrightarrow \alpha$ in the second term on the right hand side. Since δ is a dummy index we may relabel it as $\delta \to \beta$, yielding

$$g_{\alpha\gamma}\ddot{x}^{\alpha} = \frac{1}{2}\left(g_{\alpha\beta,\gamma} - g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}\right)\dot{x}^{\alpha}\dot{x}^{\beta} \ . \tag{42}$$

Multiplying both sides by $g^{\gamma\mu}$, using the identity $g_{\alpha\gamma}g^{\gamma\mu} = \delta^{\mu}_{\alpha}$ and bringing all terms to the left hand side we obtain

$$\ddot{x}^{\mu} + \frac{1}{2}g^{\gamma\mu}\left(g_{\alpha\gamma,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}\right)\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \ . \tag{43}$$

It is straightforward to confirm that the term multiplying $\dot{x}^{\alpha}\dot{x}^{\beta}$ is precisely ${\Gamma^{\mu}}_{\beta\alpha} = {\Gamma^{\mu}}_{\alpha\beta}$ and thus we obtain

$$\ddot{x}^{\mu} + {\Gamma^{\mu}}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0 \ , \tag{44}$$

which is the geodesic equation of motion, as required.

Exercise 3

Optional: Using the Einstein-Hilbert action

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\,R \ , \tag{45}$$

show that the application of a variational principle $\delta\mathcal{S} = 0$ yields the Einstein field equations in vacuum, i.e.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \ . \tag{46}$$

Solution 3

First we may write

$$\delta\mathcal{S} = 0 \iff \delta\int \mathrm{d}^4x\sqrt{-g}\,R = 0 \ . \tag{47}$$

Now let us vary $\sqrt{-g}$, yielding

$$\delta\left(\sqrt{-g}\right) = -\frac{\delta g}{2\sqrt{-g}} \ . \tag{48}$$

Now recall from Problem Sheet 7, Exercise 3, part 4, we proved the following result:

$$\left(\ln|g|\right)_{,\alpha} = g^{\mu\nu}g_{\mu\nu,\alpha} \ . \tag{49}$$

This implies that

$$g_{,\alpha} = g\,g^{\mu\nu}g_{\mu\nu,\alpha} \ , \tag{50}$$

and thus we may write δg as

$$\begin{aligned} \delta g &= g\,g^{\mu\nu}\,\delta g_{\mu\nu} \\ &= -g\,g_{\mu\nu}\,\delta g^{\mu\nu} \ . \end{aligned} \tag{51}$$

We may now write $\delta\left(\sqrt{-g}\right)$ as:

$$\begin{aligned}\delta\left(\sqrt{-g}\right) &= \frac{g\,g_{\mu\nu}\,\delta g^{\mu\nu}}{2\sqrt{-g}} \\ &= -\frac{1}{2}\frac{(-g)}{\sqrt{-g}}\,g_{\mu\nu}\,\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\ . \end{aligned} \tag{52}$$

We must next consider the variation of the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$. We may write this as

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\ . \tag{53}$$

Substituting equations (52) and (53) into equation (47) yields:

$$\begin{aligned}\delta\int \mathrm{d}^4x\sqrt{-g}\,R &= \int \mathrm{d}^4x\left[\delta\left(\sqrt{-g}\right)R + \sqrt{-g}\,\delta R\right] \\ &= \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\,\delta g^{\mu\nu}\,R + \left(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\right)\right] \\ &= \int \mathrm{d}^4x\sqrt{-g}\left[\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R\right) + g^{\mu\nu}\,\delta R_{\mu\nu}\right] \\ &= \int \mathrm{d}^4x\sqrt{-g}\left(\delta g^{\mu\nu}\,G_{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\right) = 0\ , \end{aligned} \tag{54}$$

where $G_{\mu\nu} \equiv \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,R\right)$ is the Einstein tensor. It is now clear that in order for us to obtain the Einstein field equations in vacuum, the second term in brackets in equation (54) must vanish.

Let us now turn our attention to the variation of the Ricci tensor, $\delta R_{\mu\nu}$. First, recall the definition of the Riemann curvature tensor

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}{}_{\nu\beta,\alpha} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\nu\alpha,\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha}\ . \tag{55}$$

Next, consider the variation of the Riemann curvature tensor:

$$\begin{aligned}\delta R^{\mu}{}_{\nu\alpha\beta} = &\quad \partial_\alpha\left(\delta\Gamma^{\mu}{}_{\nu\beta}\right) + \left(\delta\Gamma^{\mu}{}_{\rho\alpha}\right)\Gamma^{\rho}{}_{\nu\beta} + \Gamma^{\mu}{}_{\rho\alpha}\left(\delta\Gamma^{\rho}{}_{\nu\beta}\right) \\ &- \partial_\beta\left(\delta\Gamma^{\mu}{}_{\nu\alpha}\right) - \left(\delta\Gamma^{\mu}{}_{\rho\beta}\right)\Gamma^{\rho}{}_{\nu\alpha} - \Gamma^{\mu}{}_{\rho\beta}\left(\delta\Gamma^{\rho}{}_{\nu\alpha}\right)\ . \end{aligned} \tag{56}$$

This expression can be written much more succinctly in terms of covariant derivatives. The first and fourth terms contains a partial derivative, so we consider the following:

$$\nabla_\alpha\left(\delta\Gamma^{\mu}{}_{\nu\beta}\right) = \partial_\alpha\left(\delta\Gamma^{\mu}{}_{\nu\beta}\right) + \Gamma^{\mu}{}_{\alpha\rho}\left(\delta\Gamma^{\rho}{}_{\nu\beta}\right) - \Gamma^{\rho}{}_{\alpha\nu}\left(\delta\Gamma^{\mu}{}_{\rho\beta}\right) - \Gamma^{\rho}_{\alpha\beta}\left(\delta\Gamma^{\mu}{}_{\nu\rho}\right)\ , \tag{57}$$

$$\nabla_\beta\left(\delta\Gamma^{\mu}{}_{\nu\alpha}\right) = \partial_\beta\left(\delta\Gamma^{\mu}{}_{\nu\alpha}\right) + \Gamma^{\mu}{}_{\beta\rho}\left(\delta\Gamma^{\rho}{}_{\nu\alpha}\right) - \Gamma^{\rho}{}_{\beta\nu}\left(\delta\Gamma^{\mu}{}_{\rho\alpha}\right) - \Gamma^{\rho}_{\alpha\beta}\left(\delta\Gamma^{\mu}{}_{\nu\rho}\right)\ . \tag{58}$$

It immediately follows that the difference between equations (57) and (58) enables equation (56) to be written as

$$\delta R^{\mu}{}_{\nu\alpha\beta} = \nabla_\alpha\left(\delta\Gamma^{\mu}{}_{\nu\beta}\right) - \nabla_\beta\left(\delta\Gamma^{\mu}{}_{\nu\alpha}\right)\ . \tag{59}$$

We may now calculate $\delta R_{\mu\nu}$ as follows:

$$\begin{aligned}\delta R_{\nu\beta} &= \delta R^{\alpha}{}_{\nu\alpha\beta} \\ &= \nabla_\alpha\left(\delta\Gamma^{\alpha}{}_{\nu\beta}\right) - \nabla_\beta\left(\delta\Gamma^{\alpha}{}_{\nu\alpha}\right)\ , \end{aligned} \tag{60}$$

and thus we obtain upon relabelling indices ($\beta \leftrightarrow \nu$ followed by $\beta \to \mu$):

$$\delta R_{\mu\nu} = \nabla_\alpha \left(\delta\Gamma^\alpha{}_{\mu\nu}\right) - \nabla_\nu \left(\delta\Gamma^\alpha{}_{\mu\alpha}\right) \ . \tag{61}$$

We may now write the second term in brackets in equation (54) as:

$$\begin{aligned} g^{\mu\nu}\,\delta R_{\mu\nu} &= \nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu}\right) - \nabla_\nu \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\alpha}\right) \\ &= \nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu}\right) \ , \end{aligned} \tag{62}$$

where we have let $\alpha \leftrightarrow \nu$ in the second term. We may now write the second term in equation (54) as

$$\int \mathrm{d}^4x \sqrt{-g}\, g^{\mu\nu}\,\delta R_{\mu\nu} = \int \mathrm{d}^4x \sqrt{-g}\,\nabla_\alpha \left(g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu}\right) \ . \tag{63}$$

To proceed further, recall Problem Sheet 7, Exercise 3, part 5, where we proved the following identity:

$$A^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\,A^\alpha\right)_{,\alpha} \ . \tag{64}$$

We may define A^α from equation (63) as

$$A^\alpha = g^{\mu\nu}\delta\Gamma^\alpha{}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^\nu{}_{\mu\nu} \ , \tag{65}$$

which enables us to rewrite equation (63) as

$$\begin{aligned} \int \mathrm{d}^4x \sqrt{-g}\, g^{\mu\nu}\,\delta R_{\mu\nu} &= \int \mathrm{d}^4x\, \partial_\alpha \left(\sqrt{-g}\,A^\alpha\right) \\ &= 0 \ , \end{aligned} \tag{66}$$

since this is a surface integral, yielding a constant boundary term, and by Stokes's Theorem vanishes. We may finally write

$$\delta\mathcal{S} = \int \mathrm{d}^4x \sqrt{-g}\,\delta g^{\mu\nu}\, G_{\mu\nu} = 0 \ , \tag{67}$$

and so we may conclude that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = 0 \ , \tag{68}$$

i.e. the Einstein field equations in vacuum, as required.

General Relativity: Solutions to exercises in Lecture XIV

January 29, 2018

Exercise 1

Using the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \ , \tag{1}$$

where $\kappa = -1,\ 0,\ 1$. Compute:

- the non-zero Christoffel symbols
- the non-zero components of the Ricci tensor
- the expression for the Ricci scalar

Solution 1

- The first part of the question asks us to calculate the non-zero Christoffel symbol components of the FLRW metric. Let us begin by writing the Lagrangian for the FLRW metric:

$$\mathcal{L} = \frac{1}{2}\left(-t'^2 + \frac{a^2}{1-\kappa r^2}\,r'^2 + a^2 r^2\,\theta'^2 + a^2 r^2 \sin^2\theta\,\phi'^2\right) \ , \tag{2}$$

where primed quantities ($'$) denote differentiation with respect to the affine parameter, λ. We have also written $a \equiv a(t)$ for brevity. Next we employ the Euler-Lagrange equations, which may be written as:

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} - \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial x'^\alpha}\right) = 0 \ . \tag{3}$$

The Euler-Lagrange equations are equivalent to the geodesic equations of motion (see Lecture XIII, exercise 2) and so we can read off the Christoffel symbol components directly. First, we consider the t-component:

$$\frac{\partial \mathcal{L}}{\partial t} = a\,\dot{a}\left(\frac{r'^2}{1-\kappa r^2} + r^2\,\theta'^2 + r^2\sin^2\theta\,\phi'^2\right) \ , \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial t'} = -t' \ , \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial t'}\right) = -t'' \ , \tag{6}$$

where an overdot ($\dot{}$) denotes differentiation with respect to t. We immediately obtain:

$$t'' = -\frac{a\,\dot{a}}{1-\kappa r^2}\,r'^2 - a\,\dot{a}\left(r^2\,\theta'^2 + r^2\sin^2\theta\,\phi'^2\right)\ . \tag{7}$$

We may now read off the Christoffel symbol components directly, obtaining:

$$\Gamma^t{}_{rr} = \frac{a\dot{a}}{1-\kappa r^2}\ , \tag{8}$$
$$\Gamma^t{}_{\theta\theta} = a\,\dot{a}\,r^2\ , \tag{9}$$
$$\Gamma^t{}_{\phi\phi} = a\,\dot{a}\,r^2\sin^2\theta\ . \tag{10}$$

Next, we consider the r-component of the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial r} = a^2 r\left[\frac{\kappa}{\left(1-\kappa r^2\right)^2} + \theta'^2 + \sin^2\theta\,\phi'^2\right]\ , \tag{11}$$
$$\frac{\partial\mathcal{L}}{\partial r'} = \frac{a^2}{1-\kappa r^2}\,r'\ , \tag{12}$$
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial r'}\right) = \frac{2a\,\dot{a}}{1-\kappa r^2}t'\,r' + \frac{\kappa r}{\left(1-\kappa r^2\right)^2}r'^2 + \frac{a^2}{1-\kappa r^2}r''\ . \tag{13}$$

We thus obtain the geodesic equation of motion as:

$$r'' = -2\frac{\dot{a}}{a}t'\,r' - \frac{\kappa r}{1-\kappa r^2}r'^2 + r(1-\kappa r^2)\theta'^2 + r^2\sin^2\theta(1-\kappa r^2\phi'^2)\ , \tag{14}$$

from which the Christoffel symbols are immediately obtained as:

$$\Gamma^r{}_{rt} = \Gamma^r{}_{tr} = \frac{\dot{a}}{a}\ , \tag{15}$$
$$\Gamma^r{}_{rr} = \frac{\kappa r}{1-\kappa r^2}\ , \tag{16}$$
$$\Gamma^r{}_{\theta\theta} = -r(1-\kappa r^2)\ , \tag{17}$$
$$\Gamma^r{}_{\phi\phi} = -r^2\sin^2\theta(1-\kappa r^2)\ . \tag{18}$$

We now consider the θ-component of the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\theta} = a^2 r^2\sin\theta\cos\theta\,\phi'^2\ , \tag{19}$$
$$\frac{\partial\mathcal{L}}{\partial\theta'} = a^2 r^2\,\theta'\ , \tag{20}$$
$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial\mathcal{L}}{\partial\theta'}\right) = 2\,a\,\dot{a}\,r^2 t'\,\theta' + 2a^2 r\,r'\,\theta' + a^2 r^2\,\theta''\ . \tag{21}$$

This gives the geodesic equation of motion for θ as:

$$\theta'' = -2\frac{\dot{a}}{a}\,t'\,\theta' - \frac{2}{r}\,r'\,\theta' + \sin\theta\cos\theta\,\phi'^2\ , \tag{22}$$

from which the Christoffel symbols immediately follow as:

$$\begin{aligned}
{\Gamma^{\theta}}_{t\theta} = {\Gamma^{\theta}}_{\theta t} &= \frac{\dot{a}}{a} \;, && (23)\\
{\Gamma^{\theta}}_{r\theta} = {\Gamma^{\theta}}_{\theta r} &= \frac{1}{r} \;, && (24)\\
{\Gamma^{\theta}}_{\phi\phi} &= -\sin\theta\cos\theta \;. && (25)
\end{aligned}$$

Finally, we consider the ϕ-component of the Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= 0 \;, && (26)\\
\frac{\partial \mathcal{L}}{\partial \phi'} &= a^2 r^2 \sin^2\theta\, \phi'\, \theta' \;, && (27)\\
\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) &= 2a\,\dot{a}\,r^2 \sin^2\theta\, t'\,\phi' + 2a^2 r\, \sin^2\theta\, r'\,\phi' + 2a^2r^2 \sin\theta\cos\theta\,\theta'\,\phi' + a^2r^2\sin^2\theta\,\phi'' , && (28)
\end{aligned}$$

from which we obtain the geodesic equation of motion for ϕ as:

$$\phi'' = -2\frac{\dot{a}}{a}\, t'\,\phi' - \frac{2}{r}\, r'\,\phi' - 2\mathrm{cot}\theta\,\theta'\,\phi' \;. \tag{29}$$

Thus the final non-zero Christoffel symbols read:

$$\begin{aligned}
{\Gamma^{\phi}}_{t\phi} = {\Gamma^{\phi}}_{\phi t} &= \frac{\dot{a}}{a} \;, && (30)\\
{\Gamma^{\phi}}_{r\phi} = {\Gamma^{\phi}}_{\phi r} &= \frac{1}{r} \;, && (31)\\
{\Gamma^{\phi}}_{\theta\phi} = {\Gamma^{\phi}}_{\phi\theta} &= \mathrm{cot}\theta \;. && (32)
\end{aligned}$$

- For the second part of the question, recall the definition of the Ricci tensor, which is the contraction of the Riemann curvature tensor over the first and third indices. This may be written as:

$$\begin{aligned}
R_{\mu\nu} &= {R^{\alpha}}_{\mu\alpha\nu}\\
&= {\Gamma^{\alpha}}_{\mu\nu,\alpha} + {\Gamma^{\rho}}_{\mu\nu}{\Gamma^{\alpha}}_{\rho\alpha} - {\Gamma^{\alpha}}_{\mu\alpha,\nu} - {\Gamma^{\rho}}_{\mu\alpha}{\Gamma^{\alpha}}_{\rho\nu} \;. && (33)
\end{aligned}$$

The FLRW metric is spherically symmetric and possesses no off-diagonal terms, i.e. $g_{\mu\nu} = 0$ if $\mu \neq \nu$. Let us consider the four terms in the definition of the Ricci tensor for $\mu \neq \nu$.

In this case ${\Gamma^{\alpha}}_{\mu\nu,\alpha} = 0\;\forall\;\mu \neq \nu$ (see the Christoffel symbol components). The third term ${\Gamma^{\alpha}}_{\mu\alpha,\nu} = 0$ also, since ${\Gamma^{\alpha}}_{\mu\alpha} \propto f\left(x^{\mu}\right)$ and thus ${\Gamma^{\alpha}}_{\mu\alpha,\nu} = 0$ since $\mu \neq \nu$.

For a spherically symmetric metric ${\Gamma^{\alpha}}_{\mu\nu} = 0$ if $\alpha \neq \mu \neq \nu$, which follows from the definition of the Christoffel symbols. Using this, it may be shown that for all six independent combinations of (μ, ν), with $\mu \neq \nu$, that ${\Gamma^{\rho}}_{\mu\nu}{\Gamma^{\alpha}}_{\rho\alpha} - {\Gamma^{\rho}}_{\mu\alpha}{\Gamma^{\alpha}}_{\rho\nu} = 0$. We may thus conclude that $R_{\mu\nu} = 0$ for $\mu \neq \nu$.

Let us consider the diagonal components of the Ricci tensor term-by-term. First the tt component:

$$R_{tt} = \cancelto{0}{{\Gamma^{\alpha}}_{tt,\alpha}} + \cancelto{0}{{\Gamma^{\rho}}_{tt}}{\Gamma^{\alpha}}_{\rho\alpha} - {\Gamma^{\alpha}}_{t\alpha,t} - {\Gamma^{\rho}}_{t\alpha}{\Gamma^{\alpha}}_{\rho t} \;. \tag{34}$$

For the third and fourth terms we obtain:

$$\begin{aligned}
{\Gamma^{\alpha}}_{t\alpha,t} &= {\Gamma^{i}}_{ti,t} \\
&= 3\left(\frac{a\,\ddot{a}-\dot{a}^2}{a^2}\right) \;, \qquad (35) \\
{\Gamma^{\rho}}_{t\alpha}{\Gamma^{\alpha}}_{\rho t} &= {\Gamma^{\rho}}_{ti}{\Gamma^{i}}_{\rho t} \\
&= \left({\Gamma^{i}}_{ti}\right)^2 \text{ (only } \rho = i \text{ gives a nonzero result)} \\
&= 3\frac{\dot{a}^2}{a^2} \;, \qquad (36)
\end{aligned}$$

where the index i denotes spatial co-ordinates (r, θ, ϕ). Thus we immediately find:

$$R_{tt} = -3\frac{\ddot{a}}{a} \;. \qquad (37)$$

We next consider the rr component of the Ricci tensor:

$$R_{rr} = {\Gamma^{\alpha}}_{rr,\alpha} + {\Gamma^{\rho}}_{rr}{\Gamma^{\alpha}}_{\rho\alpha} - {\Gamma^{\alpha}}_{r\alpha,r} - {\Gamma^{\rho}}_{r\alpha}{\Gamma^{\alpha}}_{\rho r} \;. \qquad (38)$$

For the first term:

$$\begin{aligned}
{\Gamma^{\alpha}}_{rr,\alpha} &= {\Gamma^{t}}_{rr,t} + {\Gamma^{r}}_{rr,r} \\
&= \frac{\dot{a}^2 + a\,\ddot{a} + \kappa}{1-\kappa r^2} + \frac{2\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} \;. \qquad (39)
\end{aligned}$$

For the second term:

$$\begin{aligned}
{\Gamma^{\rho}}_{rr}{\Gamma^{\alpha}}_{\rho\alpha} &= {\Gamma^{t}}_{rr}{\Gamma^{\alpha}}_{t\alpha} \\
&= {\Gamma^{t}}_{rr}\left({\Gamma^{i}}_{ti}\right) + {\Gamma^{r}}_{rr}\left({\Gamma^{i}}_{ri}\right) \\
&= \frac{3\dot{a}^2 + 2\kappa}{1-\kappa r^2} + \frac{\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} \;. \qquad (40)
\end{aligned}$$

For the third term:

$$\begin{aligned}
{\Gamma^{\alpha}}_{r\alpha,r} &= {\Gamma^{i}}_{ri,r} \\
&= \frac{2\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} + \frac{\kappa}{1-\kappa r^2} - \frac{2}{r^2} \;. \qquad (41)
\end{aligned}$$

For the fourth term:

$$\begin{aligned}
{\Gamma^{\rho}}_{r\alpha}{\Gamma^{\alpha}}_{\rho r} &= {\Gamma^{\rho}}_{rt}{\Gamma^{t}}_{\rho r} + {\Gamma^{\rho}}_{ri}{\Gamma^{i}}_{\rho r} \\
&= {\Gamma^{r}}_{rt}{\Gamma^{t}}_{rr} + {\Gamma^{t}}_{ri}{\Gamma^{i}}_{tr} + {\Gamma^{i}}_{ri}{\Gamma^{i}}_{ir} \\
&= 2{\Gamma^{r}}_{rt}{\Gamma^{t}}_{rr} + \left({\Gamma^{i}}_{ri}\right)^2 \\
&= \frac{\kappa^2 r^2}{\left(1-\kappa r^2\right)^2} + \frac{2\dot{a}^2}{1-\kappa r^2} + \frac{2}{r^2} \;. \qquad (42)
\end{aligned}$$

We thus obtain:

$$R_{rr} = \frac{2\kappa + 2\dot{a}^2 + a\,\ddot{a}}{1-\kappa r^2} \;. \qquad (43)$$

The $\theta\theta$ component of the Ricci tensor yields:

$$R_{\theta\theta} = \Gamma^{\alpha}{}_{\theta\theta,\alpha} + \Gamma^{\rho}{}_{\theta\theta}\Gamma^{\alpha}{}_{\rho\alpha} - \Gamma^{\alpha}{}_{\theta\alpha,\theta} - \Gamma^{\rho}{}_{\theta\alpha}\Gamma^{\alpha}{}_{\rho\theta} \ . \tag{44}$$

For the first term we obtain:

$$\begin{aligned} \Gamma^{\alpha}{}_{\theta\theta,\alpha} &= \Gamma^{t}{}_{\theta\theta,t} + \Gamma^{r}{}_{\theta\theta,r} \\ &= -1 + 3\kappa r^2 + r^2(\dot{a}^2 + a\,\ddot{a}) \ . \end{aligned} \tag{45}$$

For the second term we obtain:

$$\begin{aligned} \Gamma^{\rho}{}_{\theta\theta}\Gamma^{\alpha}{}_{\rho\alpha} &= \Gamma^{t}{}_{\theta\theta}\Gamma^{\alpha}{}_{t\alpha} + \Gamma^{r}{}_{\theta\theta}\Gamma^{\alpha}{}_{r\alpha} \\ &= \Gamma^{t}{}_{\theta\theta}\Gamma^{i}{}_{ti} + \Gamma^{r}{}_{\theta\theta}\Gamma^{i}{}_{ri} \\ &= 3\dot{a}^2 r^2 + \kappa r^2 - 2 \ . \end{aligned} \tag{46}$$

For the third term we obtain:

$$\begin{aligned} \Gamma^{\alpha}{}_{\theta\alpha,\theta} &= \Gamma^{\phi}{}_{\theta\phi,\theta} \\ &= -\mathrm{cosec}^2\theta \ . \end{aligned} \tag{47}$$

For the fourth term we obtain:

$$\begin{aligned} \Gamma^{\rho}{}_{\theta\alpha}\Gamma^{\alpha}{}_{\rho\theta} &= \Gamma^{\rho}{}_{\theta t}\Gamma^{t}{}_{\rho\theta} + \Gamma^{\rho}{}_{\theta i}\Gamma^{i}{}_{\rho\theta} \\ &= \Gamma^{\rho}{}_{\theta t}\Gamma^{t}{}_{\rho\theta} + \Gamma^{t}{}_{\theta i}\Gamma^{i}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{t}{}_{\theta i}\Gamma^{i}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= \Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{t}{}_{\theta\theta}\Gamma^{\theta}{}_{t\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{j}{}_{\theta i}\Gamma^{i}{}_{j\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + \Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} + \Gamma^{\theta}{}_{\theta r}\Gamma^{r}{}_{\theta\theta} + \Gamma^{\phi}{}_{\theta\phi}\Gamma^{\phi}{}_{\phi\theta} \\ &= 2\Gamma^{\theta}{}_{\theta t}\Gamma^{t}{}_{\theta\theta} + 2\Gamma^{r}{}_{\theta\theta}\Gamma^{\theta}{}_{r\theta} + \left(\Gamma^{\phi}{}_{\theta\phi}\right)^2 \\ &= 2\dot{a}^2 r^2 + 2\kappa r^2 - 2 + \cot^2\theta \ . \end{aligned} \tag{48}$$

We thus obtain:

$$R_{\theta\theta} = r^2\left(2\kappa + 2\dot{a}^2 + a\,\ddot{a}\right) \ . \tag{49}$$

Finally, we consider the $\phi\phi$ component of the Ricci tensor:

$$R_{\phi\phi} = \Gamma^{\alpha}{}_{\phi\phi,\alpha} + \Gamma^{\rho}{}_{\phi\phi}\Gamma^{\alpha}{}_{\rho\alpha} - \cancelto{0}{\Gamma^{\alpha}{}_{\phi\alpha,\phi}} - \Gamma^{\rho}{}_{\phi\alpha}\Gamma^{\alpha}{}_{\rho\phi} \ . \tag{50}$$

The first term gives:

$$\begin{aligned} \Gamma^{\alpha}{}_{\phi\phi,\alpha} &= \Gamma^{i}{}_{\phi\phi,i} \\ &= -\cos^2\theta + \left(\dot{a}^2 + a\,\ddot{a} + 3\kappa\right) r^2 \sin^2\theta \ . \end{aligned} \tag{51}$$

The second term gives:

$$\begin{aligned} \Gamma^{\rho}{}_{\phi\phi}\Gamma^{\alpha}{}_{\rho\alpha} &= \Gamma^{t}{}_{\phi\phi}\Gamma^{\alpha}{}_{t\alpha} + \Gamma^{r}{}_{\phi\phi}\Gamma^{\alpha}{}_{r\alpha} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\alpha}{}_{\theta\alpha} \\ &= \Gamma^{t}{}_{\phi\phi}\Gamma^{i}{}_{ti} + \Gamma^{r}{}_{\phi\phi}\Gamma^{i}{}_{ri} + \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\phi} \\ &= \left(3\dot{a}^2 + \kappa\right) r^2 \sin^2\theta - 1 - \sin^2\theta \ . \end{aligned} \tag{52}$$

The fourth term gives:

$$\begin{aligned}
{\Gamma^\rho}_{\phi\alpha}{\Gamma^\alpha}_{\rho\phi} &= {\Gamma^\rho}_{\phi t}{\Gamma^t}_{\rho\phi} + {\Gamma^\rho}_{\phi r}{\Gamma^r}_{\rho\phi} + {\Gamma^\rho}_{\phi\theta}{\Gamma^\theta}_{\rho\phi} + {\Gamma^\rho}_{\phi\phi}{\Gamma^\phi}_{\rho\phi} \\
&= {\Gamma^\phi}_{\phi t}{\Gamma^t}_{\phi\phi} + {\Gamma^\phi}_{\phi r}{\Gamma^r}_{\phi\phi} + {\Gamma^\phi}_{\phi\theta}{\Gamma^\theta}_{\phi\phi} + \left({\Gamma^t}_{\phi\phi}{\Gamma^\phi}_{t\phi} + {\Gamma^r}_{\phi\phi}{\Gamma^\phi}_{r\phi} + {\Gamma^\theta}_{\phi\phi}{\Gamma^\phi}_{\theta\phi}\right) \\
&= 2\left({\Gamma^\phi}_{\phi t}{\Gamma^t}_{\phi\phi} + {\Gamma^\phi}_{\phi r}{\Gamma^r}_{\phi\phi} + {\Gamma^\phi}_{\phi\theta}{\Gamma^\theta}_{\phi\phi}\right) \\
&= \left(2\dot{a}^2 + 2\kappa\right) r^2 \sin^2\theta - 2 \ . \qquad (53)
\end{aligned}$$

Thus we obtain:

$$R_{\phi\phi} = r^2 \sin^2\theta \left(2\kappa + 2\dot{a}^2 + a\,\ddot{a}\right) \ . \qquad (54)$$

Defining $\mathcal{A} \equiv 2\kappa + 2\dot{a}^2 + a\,\ddot{a}$ we may write the non-zero Ricci tensor components more succinctly as:

$$\begin{aligned}
R_{tt} &= -3\frac{\ddot{a}}{a} \ , \qquad (55) \\
R_{ii} &= g_{ii}\frac{\mathcal{A}}{a^2} \ . \qquad (56)
\end{aligned}$$

- For the third part of the question we are asked to calculate the Ricci scalar. This follows straightforwardly from equations (55)–(56):

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} \\
&= g^{tt}R_{tt} + g^{ii}R_{ii} \\
&= 3\frac{\ddot{a}}{a} + g^{ii}g_{ii}\frac{\mathcal{A}}{a^2} \\
&= 3\frac{\ddot{a}}{a} + 3\frac{\mathcal{A}}{a^2} \\
&= \frac{6}{a^2}\left(\kappa + \dot{a}^2 + a\,\ddot{a}\right) \ . \qquad (57)
\end{aligned}$$

Exercise 2

Exploiting the results of the previous exercise, use the Einstein equations for the FLRW metric to derive the Friedmann equations. For simplicity set $\Lambda = 0$.

Solution 2

Since Exercise 3 requires $\Lambda > 0$ we will also assume this in the following solution. The Einstein field equations may be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \qquad (58)$$

where

$$T_{\mu\nu} = (e+p)u_\mu u_\nu + p g_{\mu\nu} \ . \qquad (59)$$

We are in the comoving frame of the fluid, where $u^\alpha = (1, \underline{0})$ and $u_\alpha = (-1, \underline{0})$, and therefore:

$$\begin{aligned}
T_{tt} &= e \ , \qquad (60) \\
T_{ii} &= p g_{ii} \ . \qquad (61)
\end{aligned}$$

Considering the tt component of the Einstein field equations we obtain:

$$\begin{aligned} & R_{tt} - \frac{1}{2} g_{tt} R + \Lambda g_{tt} = 8\pi T_{tt} \\ \Longrightarrow \quad & -3\frac{\ddot{a}}{a} + \frac{3}{a^2}\left(\kappa + \dot{a}^2 + a\,\ddot{a}\right) - \Lambda = 8\pi e \\ \Longrightarrow \quad & \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\left(8\pi e + \Lambda\right) - \frac{\kappa}{a^2}\ , \end{aligned} \tag{62}$$

which is the first Friedmann equation. We now consider the spatial component of the Einstein field equations:

$$\begin{aligned} & R_{ii} - \frac{1}{2} g_{ii} R + \Lambda g_{ii} = 8\pi T_{ii} \\ \Longrightarrow \quad & g_{ii}\frac{\mathcal{A}}{a^2} + g_{ii}\left(-\frac{1}{2}R + \Lambda - 8\pi p\right) = 0 \\ \Longrightarrow \quad & -2\frac{\ddot{a}}{a} - \frac{\kappa}{a^2} - \frac{\dot{a}^2}{a^2} + \Lambda - 8\pi p = 0 \qquad \text{(use equation (62))} \\ \Longrightarrow \quad & \frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(e + 3p\right) + \frac{\Lambda}{3}\ , \end{aligned} \tag{63}$$

which is the second Friedmann equation.

Exercise 3

Optional: Consider the case of an equation of state where $p = -e$ and $\Lambda > 0$. Derive the evolution equation for the scale factor. What type of universe is this?

Solution 3

In the comoving frame the stress-energy-momentum tensor of a perfect fluid may be written as:

$$\begin{aligned} T^{\mu}_{\ \nu} &= (e+p)u^{\mu}u_{\nu} + p\delta^{\mu}_{\nu} \\ &= \mathrm{diag}(-e, p, p, p)\ . \end{aligned} \tag{64}$$

The spatial component of the conservation equation ($\nabla_{\mu} T^{\mu}_{\ \nu} = 0$) trivially vanishes, implying uniform pressure. However, it is straightforward to show that the time component yields the fluid conservation equation:

$$\dot{e} + 3\frac{\dot{a}}{a}(e+p) = 0\ . \tag{65}$$

For the given equation of state, this implies that $\dot{e} = 0$ and hence $e = e_0 =$ constant. Substituting this into the second Friedmann equation we obtain:

$$\begin{aligned} & \frac{\ddot{a}}{a} = \frac{8\pi}{3}e_0 + \frac{\Lambda}{3} \\ \Longrightarrow \quad & \ddot{a} = \mathcal{C}a\ , \end{aligned} \tag{66}$$

where $\mathcal{C} \equiv (\Lambda + 8\pi e_0)/3$. Now, since $\Lambda > 0$ and $e_0 \geq 0$, then this implies that $\mathcal{C} > 0$. Integrating equation (66) directly yields:

$$a(t) = c_1 \mathrm{e}^{\sqrt{\mathcal{C}}t} + c_2 \mathrm{e}^{-\sqrt{\mathcal{C}}t} \ , \tag{67}$$

where the integration constants c_1 and c_2 may be calculated from this equation and the first Friedmann equation as:

$$c_1 + c_2 = a_0 \ , \tag{68}$$

$$\mathcal{C}a_0^2 - \kappa = \dot{a}_0 \ , \tag{69}$$

where a_0 and $\dot{a}_0$ are the initial values of $a(t)$ and $\dot{a}(t)$ at $t = 0$. Equation (67) has a minimum at:

$$t_{\text{min}} = \frac{1}{\sqrt{\mathcal{C}}} \ln \sqrt{\frac{c_2}{c_1}} \ , \tag{70}$$

and since $t \geq 0$ for the universe we know that if $c_2 > c_1$ then there exists a minimum value of $t > 0$. We assume c_1 and c_2 are both positive. Thus, if $c_1 > c_2$ the universe expands exponentially from $t = 0$. If, however, $c_2 > c_1$ then the universe contracts between $t = 0$ and t_{min}, before expanding exponentially thereafter.

General Relativity: Solutions to exercises in Lecture XV

February 5, 2018

Exercise 1

The simplest solution to the linearised Einstein equations is a plane wave of the form:

$$\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}\mathrm{e}^{i\kappa_\alpha x^\alpha}\right\} \ , \tag{1}$$

where $\Re$ denotes the real part, $\mathbf{A}$ is the "amplitude" tensor and κ is a null four-vector which satisfies $\kappa_\alpha\kappa^\alpha = 0$. In such a solution, the plane wave donated by equation (1) travels in the spatial direction $\vec{k} = (\kappa_x, \kappa_y, \kappa_z)/\kappa^0$, with frequency $\omega \equiv \kappa^0 = \left(\kappa_j\kappa^j\right)^{1/2}$. Determine the conditions such that the amplitude tensor $\mathbf{A}$ has only two linearly independent components, corresponding to the two states of polarisation of the gravitational waves.

Solution 1

$A_{\mu\nu}$ has at most 10 independent components. The solution to the linearised field equations $\Box\,\bar{h}_{\mu\nu} = 0$ is given by equation (1). Inserting this solution back into the linearised field equations yields:

$$\begin{aligned}\Box\,\bar{h}_{\mu\nu} &= \eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{h}_{\mu\nu} \\ &= \eta^{\alpha\beta}\partial_\alpha\left(i\kappa_\beta\bar{h}_{\mu\nu}\right) \\ &= -\eta^{\alpha\beta}\kappa_\alpha\kappa_\beta\bar{h}_{\mu\nu} \\ &= -\left(\kappa_\alpha\kappa^\alpha\right)\bar{h}_{\mu\nu} \ = \ 0 \ ,\end{aligned} \tag{2}$$

and thus we obtain the condition $\kappa_\alpha\kappa^\alpha = 0$. Defining $\kappa^\alpha = (\omega, \kappa^0\vec{k})$ and $\kappa_\alpha = (-\omega, \kappa^0\vec{k})$ we then immediately obtain $\omega \equiv \kappa^0 = \left(\kappa_j\kappa^j\right)^{1/2}$. Next, imposing the Lorentz (also know as de Donder) gauge $\bar{h}^{\mu\nu}{}_{,\mu} = 0$ we obtain:

$$\begin{aligned}\partial_\mu\left(A^{\mu\nu}\mathrm{e}^{i\kappa_\alpha x^\alpha}\right) &= iA^{\mu\nu}\kappa_\mu\mathrm{e}^{i\kappa_\alpha x^\alpha} \\ &= i\left(\kappa_\mu A^{\mu\nu}\right)\mathrm{e}^{i\kappa_\alpha x^\alpha} \ = \ 0 \ ,\end{aligned} \tag{3}$$

from which we obtain the condition:

$$\kappa_\mu A^{\mu\nu} = 0 \ , \tag{4}$$

i.e. the wave vector is orthogonal to $A^{\mu\nu}$. This condition constitutes a set of 4 algebraic equations and thus the number of degrees of freedom of $A_{\mu\nu}$ are reduced from 10 to 6. So far, whilst we have imposed the Lorentz gauge, we still have some remaining co-ordinate freedom. If we consider a co-ordinate transform of the form:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} \ , \tag{5}$$

then $\Box\, x'^{\mu} = 0$ if $\Box\, \xi^{\mu} = 0$, which is also a wave equation. This has the solution

$$\xi_{\mu} = B_{\mu} \mathrm{e}^{i\kappa_{\alpha} x^{\alpha}} \ , \tag{6}$$

where κ_{α} is the wave vector and B_{μ} are constant coefficients. The remaining co-ordinate freedom allows us to transform from $A_{\mu\nu} \to A'_{\mu\nu}$ such that:

- $A'_{\ 0\nu} = 0$ (wave amplitude transverse to its propagation direction)
- $A'^{\mu}_{\ \ \mu} = 0$ (traceless)

The choice of gauge sets $u^{\mu} A'_{\ \mu\nu} = 0$ for constant and timelike u^{μ}. The choice of Lorentz frame fixes u^{μ} to point along the time axis. Starting from the metric perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}\left([h_{\mu\nu}]^2\right) \ , \tag{7}$$

in the new co-ordinate system we obtain:

$$g'_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ , \tag{8}$$

from which we immediately relate the metric perturbations between the two co-ordinate systems as:

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \ . \tag{9}$$

Contracting both sides of this equation with $\eta^{\mu\nu}$ gives:

$$\begin{aligned} h' &= h - \xi^{\mu}_{\ \ \mu} - \xi^{\nu}_{\ \ \nu} \\ &= h - 2\,\xi^{\alpha}_{\ \ \alpha} \ . \end{aligned} \tag{10}$$

Now consider the trace-reversed part of $h'_{\mu\nu}$, namely $\bar{h}'_{\mu\nu}$, which may be simplified as follows:

$$\begin{aligned} h'_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' \\ &= h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{1}{2}\eta_{\mu\nu}\left(h - 2\,\xi^{\alpha}_{\ \ \alpha}\right) \\ &= h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\xi^{\alpha}_{\ \ \alpha} \\ &= \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\,\xi^{\alpha}_{\ \ \alpha} \ . \end{aligned} \tag{11}$$

Substituting our solution $\bar{h}_{\mu\nu} = A_{\mu\nu} \mathrm{e}^{i\kappa_{\alpha} x^{\alpha}}$ for the field equations and our solution $\xi_{\mu} = B_{\mu} \mathrm{e}^{i\kappa_{\alpha} x^{\alpha}}$ for the transformation between frames into equation (11) yields:

$$A'_{\mu\nu} = A_{\mu\nu} - i\kappa_{\nu} B_{\mu} - i\kappa_{\mu} B_{\nu} + i\eta_{\mu\nu}\kappa_{\alpha} B^{\alpha} \ . \tag{12}$$

We may now use the above equation to determine the components of $\mathbf{B}$ in terms of $A_{\mu\nu}$. Imposing the traceless condition implies contracting equation (12) with $\eta^{\mu\nu}$, yielding:

$$\begin{aligned} A'^{\mu}_{\ \ \mu} &= A^{\mu}_{\ \ \mu} - i\kappa_{\nu} B^{\nu} - i\kappa_{\mu} B^{\mu} + 4i\kappa_{\alpha} B^{\alpha} \\ &= A^{\mu}_{\ \ \mu} + 2i\kappa_{\mu} B^{\mu} \ = \ 0 \ , \end{aligned} \tag{13}$$

which gives the condition:

$$\kappa_{\mu} B^{\mu} = \frac{i}{2} A^{\mu}_{\ \ \mu} \ . \tag{14}$$

We next impose the transverse condition. Let us consider the temporal and spatial parts separately.

- For $\nu = 0$ we obtain:

$$A'_{00} = A_{00} - 2i\kappa_0 B_0 - i\kappa_\alpha B^\alpha = 0 \ . \tag{15}$$

  However, we have previously derived equation (14) which upon substitution and simplification gives the temporal component of $\mathbf{B}$ as:

$$B_0 = -\frac{i}{2\kappa_0}\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right) \ . \tag{16}$$

- For $\nu = j$ we obtain:

$$\begin{aligned} A'_{0j} &= A_{0j} - i\kappa_0 B_j - i\kappa_j B_0 \\ &= A_{0j} - i\kappa_0 B_j - \frac{\kappa_j}{2\kappa_0}\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right) \ . \end{aligned} \tag{17}$$

  From this we can solve for B_j, which upon simplification yields:

$$B_j = \frac{i}{2\kappa_0^2}\left[-2\kappa_0 A_{0,j} + \kappa_j\left(A_{00} + \frac{1}{2}A^\alpha{}_\alpha\right)\right] \ . \tag{18}$$

We now have the four constant coefficients for B_μ. We know that equations (17) and (18) satisfy the transverse condition and therefore $A'_{\mu\nu} \leftrightarrow A_{\mu\nu}$. The traceless condition implies 1 condition on the number of independent components of the amplitude tensor. For $\nu = 0$ the transverse condition implies $\kappa_\mu A^{\mu\nu} = 0$ which is redundant as we have already considered this. As such A'_{0j} (from the transverse condition) yields 3 conditions for $A'_{\mu\nu}$ (and therefore $A_{\mu\nu}$). Thus we conclude that the TT gauge gives us a further 4 constraints and so the number of linearly independent components of $A_{\mu\nu}$ is reduced from 6 to 2. Therefore the gravitational wave only has 2 independent states of polarisation, as required.

As an example, consider a gravitational wave travelling in the positive z-direction, where $k^\mu = (\omega, 0, 0, \kappa^z) \equiv (\omega, 0, 0, \omega)$. For such a null vector the conditions $k^\mu A_{\mu\nu} = 0$ and $A_{0\nu} = 0$ imply that $A_{3\nu} = 0$ also. Thus the only non-zero components are A_{11}, A_{12}, A_{21} and A_{22}. However, using the traceless condition $A^\mu{}_\mu = 0$ we also obtain $A_{22} = -A_{11}$. Finally, we know that by symmetry $A_{12} = A_{21}$ and thus we may write the amplitude tensor for such a gravitational wave as:

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ , \tag{19}$$

which only possesses 2 linearly independent components. It is important to note the that in the TT gauge we have $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$.

Exercise 2

The gauge satisfying the requirement of the first exercise is also refereed to as the TT (or transverse-traceless) gauge. Compute the non-zero components of the Riemann tensor in this gauge.

Solution 2

Recall from Exercise 1 that we defined the transformation $x'^\mu = x^\mu + \xi^\mu$. From the consideration of nearby geodesics/particles separated by an infinitesimal distance ξ^μ (where $||\xi^\mu|| \ll 1$) one may calculate the geodesic deviation equation:

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{\nu\alpha\beta} u^\nu u^\alpha \xi^\beta \ . \qquad (20)$$

Let us calculate the RHS of equation (20) to first order in $h_{\mu\nu}$. Assuming neighbouring geodesics/particles vary slowly, we may express the four-velocity as a unit vector in the time direction plus corrections of $\mathcal{O}(h_{\mu\nu})$ and higher. Since the Riemann tensor is already first order in $h_{\mu\nu}$, corrections to u^ν can be ignored and we may set $u^\nu = (1, 0, 0, 0)$. With this the non-zero components of the geodesic deviation equation are found as

$$\frac{D^2\xi^\mu}{D\tau^2} = R^\mu{}_{00\beta}\, \xi^\beta \ . \qquad (21)$$

Since $R^\mu{}_{00\beta} \neq 0$ this implies that $R_{\mu 00\beta} \neq 0$ also. From the symmetries of the Riemann curvature tensor we obtain:

$$R_{\mu 00\beta} = R_{0\mu 0\beta} = -R_{\mu 0\beta 0} = -R_{0\mu 0\beta} \ , \qquad (22)$$

which are the only non-zero components. Thus there is only one independent component to the Riemann curvature tensor. We may write the expression for the Riemann curvature tensor as:

$$R_{\mu 00\beta} = \frac{1}{2}\left(h_{\mu\beta,00} + h_{00,\mu\beta} - h_{\mu 0,0\beta} - h_{\beta 0,0\mu}\right)^{\mathrm{TT}} , \qquad (23)$$

where the superscript TT denotes evaluation of that quantity in the TT gauge. However, we know that $h_{\mu 0} = 0$ in the TT gauge and so the last three terms in equation (23) vanish, yielding:

$$R_{\mu 00\beta} = \frac{1}{2}\bar{h}^{\mathrm{TT}}_{\mu\beta,00} \ . \qquad (24)$$

From the plane wave solution given in equation (1) we obtain:

$$\begin{aligned} \bar{h}^{\mathrm{TT}}_{\mu\beta,00} &= -\kappa_0\kappa_0 \bar{h}^{\mathrm{TT}}_{\mu\beta} \\ &= -\omega^2 \bar{h}^{\mathrm{TT}}_{\mu\beta} \end{aligned} \qquad (25)$$

and therefore:

$$R_{\mu 00\beta} = -\frac{1}{2}\omega^2 \bar{h}^{\mathrm{TT}}_{\mu\beta} \ . \qquad (26)$$

Finally, in the TT gauge, and given our solution for $\bar{h}^{\mathrm{TT}}_{\mu\beta,00}$, we may assume $\bar{h}^{\mathrm{TT}}_{\mu\beta} \propto \mathrm{e}^{-i\omega t}$ and therefore:

$$R_{\mu 00\beta} \sim -\frac{1}{2}\omega^2 \mathrm{e}^{-i\omega t} \ . \qquad (27)$$